KEY

TO

THOMSON'S

ELEMENTARY TREATISE

ON

ALGEBRA.

BY

JAMES THOMSON, LL.D.

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF GLASGOW.

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PREFACE.

The following Publication is intended mainly for the use of the Public Teacher, and of the Student who endeavours in private to acquire a knowledge of Algebra solely by means of his own exertions. To the former it may be of service, by enabling him to detect the mistakes of his pupils, or to point out the proper plan, with less exercise of thought, and less expenditure of time, than might otherwise be required; while, to the self-taught Student, it may prevent the discouragement which always arises from failure; and even after, by his exertions, he may have succeeded in the solution of a problem, the Key may afford him additional information, which may extend his knowledge. or, what is often better, may excite new trains of thought in his mind.

Books of this kind, in consequence of the comparatively heavy expense of their publication, an expense which is not compensated for by a speedy sale, must necessarily be charged at a higher price than works of the like size, but of a more popular character. On this account, it has been the wish of the Author and the Publishers to limit the size, and consequently the price of the work; and thus to tax, as lightly as possible, those who may feel it to be their interest to use it; and for this reason the solutions of some of the simplest and easiest of the Exercises have been omitted, and those of several of the others have been given in a somewhat abridged form.

Glasgow College, May 1. 1847.

CONTENTS.

KEY

TO

THOMSON'S ALGEBRA.

COMPUTATIONS OF ALGEBRAIC EXPRESSIONS.

(ALGEBRA, p. 7.)

1. $x=\frac{125}{4+1}-\frac{8}{1}+\frac{16+12}{4+2+1}-10+3=25-8+4-10$
$+3=14.$

2. $x=\frac{25+16-1}{5+4+1}+\frac{40}{8+2}-\frac{100-80}{4+1}=4+4-4=4.$

3. $y=\frac{24+160+16}{5+4-2+1}-\frac{125-64-8+7}{32}=25-1\frac{7}{8}=23\frac{1}{8}.$

4. $y=\sqrt{(25+16-5)}-3\sqrt{(25-16-4-1)}$
$=6-3\times 2=0.$

NUMERICAL EQUATIONS OF THE FIRST DEGREE.

(ALGEBRA, p. 14.)

Exer. 1.

Since $6x+31=281-4x$, we get, by transposition,

$6x+4x=281-31$; or, by contraction, $10x=250$;

whence the answer is got by dividing by 10.

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Exer. 2.

By transposition we get

$$x+3x=319-11\text{ ; or, by contraction, }4x=308\text{ ;}$$

from which the answer is obtained by dividing by 4.

Exer. 3.

By multiplying by 5, we get

$$10x-45=360+x\text{ ;}$$

whence, by transposition and contraction,

$$10x-x=360+45,\text{ and }9x=405\text{ ;}$$

and from this the value of x is found by dividing by 9.

Exer. 4.

Here, by multiplying by 5, to get rid of the fraction, we obtain

$$5x-55=x+2+35\text{ ;}$$

whence, by transposition and contraction, we get

$$5x-x=2+35+55,\text{ and }4x=92\text{ ;}$$

and the answer is found from this by dividing by 4.

Exer. 5.

Here, by multiplying by 2 and 3 successively, we get

$$x-2=\frac{2x}{3}+2,\text{ and }3x-6=2x+6\text{ ;}$$

and hence, by transposition,

$$3x-2x=6+6\text{ ; or, by contraction, }x=12.$$

Exer. 6.

By multiplying successively by 5 and 4, we get

$$55-x=65-\frac{5x}{4},\text{ and }220-4x=260-5x\text{ ;}$$

and thence, by transposition,

$$5x-4x=260-220,\text{ or }x=40.$$

Exer. 7.

Here, by multiplying by 12 ($=4\times3=6\times2$), the least common multiple of 4 and 6, we get

$$3x+3+2x-2=96;$$

whence, by transposition and contraction,

$$3x+2x=96-3+2, \text{ and } 5x=95:$$

and, then, the value of x is obtained by dividing by 5.

Exer. 8.

Here, by multiplying by 120 ($=8\times15=12\times10=20\times6$), which is the least common multiple of the denominators, 8, 12, and 20, we get

$$15x-45+10x+90=18x+42+360;$$

whence, by transposition and contraction,

$$15x+10x-18x=42+360+45-90, \text{ and } 7x=357:$$

and from this the answer is found by dividing by 7.

Exer. 9.

The least common multiple of the denominators 3, 5, 7, 2, 4, and 6, is 420; and, by multiplying by this, we get

$$280x+336x-360x=210x+315x-350x+34020:$$

whence, by transposition, by contraction, and by dividing by 81, we obtain successively

$$280x+336x-360x-210x-315x+350x=34020;$$
$$81x=34020; \text{ and } x=420.$$

Exer. 10.

Here, by multiplying by 12, the least common multiple of 2, 3, and 4, we get

$$6x-6+4x-8-3x+9^{*}=72;$$

* Here the sign of 9 must be changed according to the principle explained in the ALGEBRA, § 33.; and the same must always be done in the case of fractions preceded by the sign *minus*, and having compound quantities as their numerators.

whence, by transposition, by contraction, and by division by 7, we get successively

$$6x+4x-3x=72+6+8-9\,;$$
$$7x=77\,;\text{ and }x=11.$$

Exer. 11.

Let the parts be x and $11-x$. Then $2x+\frac{11-x}{2}=16$; the resolution of which will give x.

Exer. 12.

Let each of the four results be denoted by x. Then, $x-1$, $x+2$, $\frac{1}{3}x$, and $4x$ will be the four parts. The sum of these is $6x+\frac{1}{3}x+1$; by putting which equal to 39, and resolving the equation, we get $x=6$; and from this the parts $x-1$, $x+2$, &c. are easily found.

Exer. 13.

Let x denote the hours between starting and meeting. Then, by multiplying x successively by $10\frac{1}{2}$ and $9\frac{1}{2}$, adding the results, putting the sum equal to 400, and resolving the equation, we get $x=20$ hours; the products of which into $10\frac{1}{2}$ and $9\frac{1}{2}$ are the respective distances in miles from London and Glasgow.

Exer. 14.

In two hours, the coach from Glasgow will have travelled 19 miles. Then, $400-19=381$, the miles still to be travelled. By proceeding as in the last question, it will be found that this space will be travelled in $19\frac{1}{20}$ hours; the product of which by $10\frac{1}{2}$ is $200\frac{1}{40}$ miles, the distance of the point of meeting from London.

Exer. 15.

Let x be the first part; then $2x$ and $60-3x$ will be the two remaining parts. Multiplying these respectively, by 12, 14, and 10, we get $12x$, $28x$, and $600-30x$, the prices in shillings,

the sum of which is $600+10x$. Putting this equal to 760, the shillings in £38 (=£30+£8), and resolving the equation, we get $x=16$, the first part; and thence we have $2x=32$, and $60-3x=12$.

Exer. 16.

Let the capital of the first dealer be x. Then $2x-100$; $2(2x-100)-100$, or $4x-300$; and $2(4x-300)-100$, or $8x-700$, will be the amounts of his property at the ends of the first, second, and third years. Putting the last equal to $2x$, and resolving the equation, we get $x=£116\frac{2}{3}$, or £116 13*s.* 4*d.*, the answer. The second answer is found by putting $8x-700$ equal to $\frac{1}{2}x$.

Exer. 17.

If x be put to denote the original capital, we shall have $2x-300$; $2(2x-300)-400$, or $4x-1000$; and $2(4x-1000)-500$, or $8x-2500$, the several amounts of his capital at the ends of the three years; and, by putting the last of these equal to 5500, we get $x=1000$, the number of pounds in his original capital.

Exer. 18.

Let the son's present age be denoted by x, and the father's will be $3x$. By taking 5 from each, we get $x-5$ and $3x-5$, their ages five years ago. Then, by putting four times the former equal to the latter, we find $x=15$; and, consequently, $3x=45$.

Exer. 19.

If x be put to denote C.'s share, A.'s will be $x+100$, and B.'s $x-50$. Adding these together, and putting the sum equal to £1000, we get $x=$£316 13*s.* 4*d.*; and consequently, $x+$£100 $=$£416 13*s.* 4*d.*, and $x-$£50 $=$ £266 13*s.* 4*d.*

Exer. 20.

Let x be the number of gallons in the cask; then $21(x+10)$ and $18(x+20)$ will each be the value of the brandy; and, putting these equal to each other, and resolving the equation, we get $x=50$.

Exer. 21.

By putting x to denote the number, we have

$$\frac{1}{2}\left(\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}\right)+20=\frac{x}{6}+\frac{x}{7}+\frac{x}{8}+\frac{x}{9}+\frac{x}{10};$$

the resolution of which equation gives $x=5040$.

Exer. 22.

If x be put to denote the less, $x+6$ will represent the greater; and, by the question, we shall have, $3x=2(x+6)+7$, or $3x=2x+12+7$; whence $x=19$, and therefore $x+6=25$.

Exer. 23.

Let x represent the required number. Then, by the question,

$$\frac{x+1}{2}+\frac{x+2}{3}=\frac{x+3}{4}+8;$$

the resolution of which equation will give $x=13$.

MULTIPLICATION.

(ALGEBRA, p. 25.)

Exer. 11.

Second power, $4x^2+12xy+9y^2$; and the answer will be found by multiplying this by $2x+3y$.

Exer. 12.

Second power, $4x^2-12xy+9y^2$; third power, $8x^3-36x^2y+54xy^2-27y^3$; and the answer will be found by taking the product of these two quantities. The fourth power is $16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$; the product of which by $2x-3y$ would also be the answer.

Exer. 13.

This exercise is easily wrought by finding the product of the first and second factors, and that of the third and fourth; and then by finding the product of the results.*

In this way the work will be as follows:

(1.)

$$\begin{array}{r}
a+b+c \\
-a+b+c \\
\hline
-a^2-ab-ac \qquad\qquad \\
ab+b^2+bc \qquad \\
ac+bc+c^2 \\
\hline
-a^2+b^2+2bc+c^2 \qquad
\end{array}$$

(2.)

$$\begin{array}{r}
a-b+c \\
a+b-c \\
\hline
a^2-ab+ac \qquad\qquad \\
ab-b^2+bc \qquad \\
-ac+bc-c^2 \\
\hline
a^2-b^2+2bc-c^2 \qquad
\end{array}$$

(3.)

$$\begin{array}{l}
-a^2+b^2+2bc+c^2 \\
\quad a^2-b^2+2bc-c^2 \\
\hline
-a^4+a^2b^2+2a^2bc+a^2c^2 \\
\qquad\qquad a^2b^2-b^4-2b^3c-b^2c^2 \\
\qquad\qquad\qquad -2a^2bc+2b^3c+4b^2c^2+2bc^3 \\
\qquad\qquad\qquad\qquad\qquad a^2c^2-b^2c^2-2bc^3-c^4 \\
\hline
-a^4+2a^2b^2-b^4+2a^2c^2+2b^2c^2-c^4.
\end{array}$$

* This exercise may be very easily wrought by means of the principle established in the ALGEBRA, § 57. In working it by that means, which the learner, when prepared for it, ought to do for his improvement, the first and second factors may be written

$$(b+c)+a \text{ and } (b+c)-a,$$

the product of which is

$$(b+c)^2-a^2, \text{ or } b^2+2bc+c^2-a^2.$$

The two remaining factors may be written

$$a-(b-c) \text{ and } a+(b-c),$$

and the product of these is

$$a^2-(b-c)^2, \text{ or } a^2-b^2+2bc-c^2.$$

These two products may be written

$$2bc-(a^2-b^2-c^2) \text{ and } 2bc+(a^2-b^2-c^2):$$

the product of which is easily found, and is the answer.

Exer. 14.*

$$
\begin{array}{l}
x^2-y^2+z^2-v^2 \\
x^2+y^2-z^2-v^2 \\
\hline
x^4-x^2y^2+x^2z^2-x^2v^2 \\
\qquad x^2y^2-y^4+y^2z^2-y^2v^2 \\
\qquad\qquad -x^2z^2+y^2z^2-z^4+z^2v^2 \\
\qquad\qquad\qquad -x^2v^2+y^2v^2-z^2v^2+v^4 \\
\hline
x^4-y^4-2x^2v^2+2y^2z^2-z^4+v^4
\end{array}
$$

Exer. 15.†

$$
\begin{array}{l}
2x-y \\
2x+y \\
\hline
4x^2-2xy \\
\qquad 2xy-y^2 \\
\hline
4x^2-y^2 \\
4x^2+y^2 \\
\hline
16x^4-4x^2y^2 \\
\qquad 4x^2y^2-y^4 \\
\hline
16x^4-y^4
\end{array}
$$

Exer. 16.‡

$$
\begin{array}{l}
1+x+x^2+x^3+x^4 \\
1+x+x^2+x^3+x^4 \\
\hline
1+x+x^2+x^3+x^4 \\
\quad x+x^2+x^3+x^4+x^5 \\
\qquad x^2+x^3+x^4+x^5+x^6 \\
\qquad\quad x^3+x^4+x^5+x^6+x^7 \\
\qquad\qquad x^4+x^5+x^6+x^7+x^8 \\
\hline
1+2x+3x^2+4x^3+5x^4+4x^5+3x^6+2x^7+x^8.
\end{array}
$$

* This is easily wrought by means of § 57., ALGEBRA, p. 44., by putting the factors under the forms,

$$x^2-v^2-(y^2-z^2) \text{ and } x^2-v^2+(y^2-z^2).$$

† This exercise is very easily wrought by means of § 57., ALGEBRA, p. 44.

‡ The work of this exercise would be effected most easily by employing only the coefficients.

Exer. 17.

1	−2	3		
4	5	−6		
4	−8	12		
	5	−10	15	
		−6	12	−18
4	−3	−4	27	−18

Exer. 18.

5	−7	−8	3	1	
7	−8				
35	−49	−56	21	7	
	−40	56	64	−24	−8
35	−89	0	85	−17	−8

Exer. 19.

1	−2	0	3			
1	0	2	−3			
1	−2	0	3			
		2	−4	0	6	
			−3	6	0	−9
1	−2	2	−4	6	6	−9

Exer. 20.

1	−4	6	−4	1			
1	−3	3	−1				
1	−4	6	−4	1			
	−3	12	−18	12	−3		
		3	−12	18	−12	3	
			−1	4	−6	4	−1
1	−7	21	−35	35	−21	7	−1

Exer. 21.

1	0	−1	2		
1	−1	2			
1	0	−1	2		
	−1	0	1	−2	
		2	0	−2	4
1	−1	1	3	−4	4

Exer. 22.

$$\begin{array}{rrrrrrr}
1 & -2 & -3 & & & & \\
1 & -2 & -3 & & & & \\
\hline
1 & -2 & -3 & & & & \\
 & -2 & 4 & 6 & & & \\
 & & -3 & 6 & 9 & & \\
\hline
1 & -4 & -2 & 12 & 9 & & \\
1 & -2 & -3 & & & & \\
\hline
1 & -4 & -2 & 12 & 9 & & \\
 & -2 & 8 & 4 & -24 & -18 & \\
 & & -3 & 12 & 6 & -36 & -27 \\
\hline
1 & -6 & 3 & 28 & -9 & -54 & -27
\end{array}$$

Exer. 23.

$$\begin{array}{rrrrr}
1 & -1 & & & \\
1 & 2 & & & \\
\hline
1 & -1 & & & \\
 & 2 & -2 & & \\
\hline
1 & 1 & -2 & & \\
1 & 4 & & & \\
\hline
1 & 1 & -2 & & \\
 & 4 & 4 & -8 & \\
\hline
1 & 5 & 2 & -8 & \\
1 & -5 & & & \\
\hline
1 & 5 & 2 & -8 & \\
 & -5 & -25 & -10 & 40 \\
\hline
1 & 0 & -23 & -18 & 40
\end{array}$$

Exer. 24.

$$\begin{array}{rrrrrrrr}
1 & -1 & 1 & -1 & 1 & -1 & & \\
1 & 1 & 1 & & & & & \\
\hline
1 & -1 & 1 & -1 & 1 & -1 & & \\
 & 1 & -1 & 1 & -1 & 1 & -1 & \\
 & & 1 & -1 & 1 & -1 & 1 & -1 \\
\hline
1 & 0 & 1 & -1 & 1 & -1 & 0 & -1
\end{array}$$

Exer. 25.

$$
\begin{array}{rrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & & \\
1 & -1 & 1 & & & & & \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & & \\
 & -1 & -1 & -1 & -1 & -1 & -1 & \\
 & & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 0 & 1 & 1 & 1 & 1 & 0 & 1
\end{array}
$$

Exer. 26.

$$
\begin{array}{rrrrrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & & & & & \\
1 & -1 & 1 & -1 & 1 & -1 & & & & & \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & & & & & \\
 & -1 & -1 & -1 & -1 & -1 & -1 & & & & \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & & & \\
 & & & -1 & -1 & -1 & -1 & -1 & -1 & & \\
 & & & & 1 & 1 & 1 & 1 & 1 & 1 & \\
 & & & & & -1 & -1 & -1 & -1 & -1 & -1 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1
\end{array}
$$

Exer. 27.

$$
\begin{array}{l}
a+2b+c \\
a-c \\
\hline
a^2+2ab+ac \\
\quad -ac-2bc-c^2 \\
\hline
a^2+2ab-2bc-c^2
\end{array}
$$

Exer. 28.

$$\begin{array}{l}
xy-1\\
xz-1\\
\hline
x^2yz-xz\\
\qquad -xy+1\\
\hline
x^2yz-xy-xz+1\\
\;\; yz\;-1\\
\hline
x^2y^2z^2-xy^2z-xyz^2+yz\\
\qquad -x^2yz+xy+xz-1\\
\hline
x^2y^2z^2-x^2yz-xy^2z-xyz^2+xy+xz+yz-1, \text{ or}\\
\qquad x^2y^2z^2-xyz(x+y+z)+xy+xz+yz-1
\end{array}$$

Exer. 29.

$$\begin{array}{l}
x^2+yz\\
y^2+xz\\
\hline
x^2y^2+y^3z\\
\qquad x^3z+xyz^2\\
\hline
x^2y^2+x^3z+y^3z+xyz^2\\
\;\; z^2+xy\\
\hline
x^2y^2z^2+x^3z^3+y^3z^3+xyz^4\\
\qquad x^3y^3+x^4yz+xy^4z+x^2y^2z^2\\
\hline
2x^2y^2z^2+x^3y^3+x^3z^3+y^3z^3+x^4yz+xy^4z+xyz^4, \text{ or}\\
\qquad 2x^2y^2z^2+x^3y^3+x^3z^3+y^3z^3+xyz(x^3+y^3+z^3)
\end{array}$$

Exer. 30.

$$\begin{array}{l}
a^2+b^2+c^2-ab-ac-bc\\
a+b+c\\
\hline
a^3+ab^2+ac^2-a^2b-a^2c-abc\\
\qquad a^2b+b^3+bc^2-ab^2-abc-b^2c\\
\qquad\qquad a^2c+b^2c+c^3-abc-ac^2-bc^2\\
\hline
a^3+b^3+c^3-3abc
\end{array}$$

DIVISION.

(ALGEBRA, p. 34—40.)

Exer. 6.

$$\begin{array}{rrr|rr} 0 & -13 & 12 & -3 & 4 \\ \underline{-3} & 4 & \underline{-12} & & \\ -3 & \underline{9} & 0 & & \\ & 0 & & & \end{array}$$

Ans. x^3-3ax^2.

Exer. 7.

$$\begin{array}{rrrr|rr} 0 & -9 & 12 & -4 & 3 & -2 \\ \underline{3} & -2 & -6 & \underline{4} & & \\ 3 & \underline{9} & \underline{-6} & 0 & & \\ & -2 & 0 & & & \end{array}$$

Ans. $x^2+3xy-2y^2$.

Exer. 8.

$$\begin{array}{rrrr|rr} -6 & 5 & 12 & 4 & 3 & 2 \\ \underline{3} & 2 & -6 & \underline{-4} & & \\ -3 & \underline{-9} & \underline{-6} & 0 & & \\ & -2 & 0 & & & \end{array}$$

Ans. x^2-3x-2.

Exer. 9.

$$\begin{array}{rr|r} 0 & 1 & 1 \\ \underline{1} & \underline{1} & \\ 1 & 2 & \end{array}$$

Ans. $x+a+\dfrac{2a^2}{x-a}$.

Exer. 12.

$$\begin{array}{rrr|rr} 0 & 0 & 1 & 0 & -1 \\ \underline{0} & -1 & \underline{0} & & \\ 0 & \underline{0} & 1 & & \\ & -1 & & & \end{array}$$

Ans. $x-\dfrac{a^2x-a^3}{x^2+a^2}$.

Exer. 14.

$$\begin{array}{rrrrr|r} -3 & 1 & 1 & -3 & 1 & -1 \\ \underline{-1} & \underline{4} & \underline{-5} & \underline{4} & \underline{-1} & \\ -4 & 5 & -4 & 1 & 0 & \end{array}$$

Ans. $x^4-4x^3+5x^2-4x+1$.

Exer. 15.

$$\begin{array}{rrrrr|r} -1 & -2 & -2 & 5 & -2 & 2 \\ \underline{2} & \underline{2} & \underline{0} & \underline{-4} & \underline{2} & \\ 1 & 0 & -2 & 1 & 0 & \end{array}$$

Ans. x^4+x^3-2x+1.

Exer. 16.

$$\begin{array}{rrrrr|r} 4 & 0 & -33 & 8 & -3 & -3 \\ & \underline{-12} & \underline{36} & \underline{-9} & \underline{3} & \\ & -12 & 3 & -1 & 0 & \end{array}$$

Ans. $4x^3-12x^2+3x-1$.

Exer. 17.

$$\begin{array}{rrrr|r} 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & -1 & 1 & \\ \hline -1 & 1 & -1 & 1 & \end{array}$$

Ans. $x^3-x^2+x-1+\dfrac{1}{x+1}$.

Exer. 18.

$$\begin{array}{rrrr|r} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & \\ \hline 1 & 1 & 1 & 1 & \end{array}$$

Ans. $x^3+x^2+x+1+\dfrac{1}{x-1}$.

Exer. 19.

$$\begin{array}{rrrr|rr} 3 & 0 & 0 & -4 & 0 & -3 \\ 0 & -3 & -9 & 9 & & \\ \hline 3 & 0 & 0 & 5 & & \\ & -3 & -9 & & & \end{array}$$

Ans. $x^2+3x-3-\dfrac{9x-5}{x^2+3}$.

Exer. 21.

$$\begin{array}{rrrrr|rrr} 7 & -26 & 50 & -74 & 35 & 3 & -5 & 7 \\ & 21 & -35 & 49 & -35 & & & \\ \hline & -5 & -15 & 25 & 0 & & & \\ & & 0 & 0 & & & & \end{array}$$

Ans. $7x-5$.

Exer. 22.

$$\begin{array}{rrrrr|r} 2 & -3 & 2 & 0 & 1 & 4 \\ & 8 & 20 & 88 & 352 & \\ \hline & 5 & 22 & 88 & 353 & \end{array}$$

Ans. $2x^3+5x^2y+22xy^2+88y^3+\dfrac{353y^4}{x-4y}$.

Exer. 23.

$$\begin{array}{rrrrr|rr} 0 & 0 & 0 & 0 & 0 & -2 & -1 \\ -2 & -1 & 2 & -3 & 4 & & \\ \hline -2 & 4 & -6 & 8 & 4 & & \\ & 3 & -4 & 5 & & & \end{array}$$

Ans. $x^3-2x^2+3x-4+\dfrac{5x+4}{x^2+2x+1}$.

Exer. 24.

$$\begin{array}{rrrrr|rr}
0 & 0 & 0 & 0 & 0 & 2 & -1 \\
2 & -1 & -2 & -3 & -4 & & \\
2 & 4 & 6 & 8 & -4 & & \\
 & 3 & 4 & 5 & & &
\end{array}$$

Ans. $x^3+2x^2+3x+4+\dfrac{5x-4}{x^2-2x+1}$.

Exer. 25.

$$\begin{array}{rrrr|rr}
0 & 0 & -8 & 7 & 3 & -2 \\
3 & -2 & -6 & -14 & & \\
3 & 9 & 21 & -7 & & \\
 & 7 & 7 & & &
\end{array}$$

Ans. $x^2+3x+7+\dfrac{7x-7}{x^2-3x+2}$.

Exer. 26.

$$\begin{array}{rrrrr|rr}
5 & -6 & 0 & 2 & -1 & 1 & -1 \\
 & 5 & -5 & 1 & 6 & & \\
 & -1 & -1 & -6 & 5 & & \\
 & & -6 & -3 & & &
\end{array}$$

Ans. $5x^2-ax-6a^2-\dfrac{3a^3x-5a^4}{x^2-ax+a^2}$.

Exer. 27.

$$\begin{array}{rrrrrr|rrr}
0 & -3 & 0 & 3 & 0 & -1 & 3 & -3 & 1 \\
3 & -3 & 1 & 3 & 3 & 1 & & & \\
3 & 9 & -9 & -9 & -3 & 0 & & & \\
 & 3 & 9 & 3 & 0 & & & & \\
 & & 1 & 0 & & & & &
\end{array}$$

Ans. $x^3+3x^2v+3xv^2+v^3$.

Exer. 28.

$$
\begin{array}{rrrrrrr|rrr}
3 & 0 & -37 & 35 & 7 & 0 & 2 & -3 & 4 & 2 \\
 & -9 & 12 & 6 & -18 & 4 & -2 & & & \\
\hline
 & -9 & 27 & -36 & 8 & -4 & 0 & & & \\
 & & 2 & -6 & 3 & 0 & & & & \\
 & & & -1 & 0 & & & & &
\end{array}
$$

Ans. $3x^3-9x^2+2x-1$.

Exer. 29.

$$
\begin{array}{l|l}
9a^3b+9a^2bc-4ab^3+4b^3c-9abc^2-9bc^3 & 3a-2b+3c \\
\underline{9a^3b-6a^2b^2+9a^2bc} & 3a^2b+2ab^2-2b^2c-3bc^2 \\
\quad 6a^2b^2-4ab^3 & \\
\quad \underline{6a^2b^2-4ab^3+6ab^2c} & \\
\qquad -6ab^2c+4b^3c-9abc^2 & \\
\qquad \underline{-6ab^2c+4b^3c-6b^2c^2} & \\
\qquad\quad -9abc^2+6b^2c^2-9bc^3 & \\
\qquad\quad \underline{-9abc^2+6b^2c^2-9bc^3} & \\
\qquad\qquad 0 &
\end{array}
$$

Exer. 30.

$$
\begin{array}{l|l}
a^3+3a^2b+3ab^2+2b^3+3b^2c+3bc^2+c^3 & a+2b+c \\
\underline{a^3+2a^2b+a^2c} & a^2+ab+b^2-ac+bc+c^2 \\
\quad a^2b+3ab^2-a^2c & \\
\quad \underline{a^2b+2ab^2+abc} & \\
\qquad ab^2-abc-a^2c+2b^3+3b^2c & \\
\qquad \underline{ab^2+2b^3+b^2c} & \\
\qquad\quad -a^2c-abc+2b^2c & \\
\qquad\quad \underline{-a^2c-2abc-ac^2} & \\
\qquad\qquad abc+2b^2c+ac^2+3bc^2 & \\
\qquad\qquad \underline{abc+2b^2c+bc^2} & \\
\qquad\qquad\quad ac^2+2bc^2+c^3 & \\
\qquad\qquad\quad \underline{ac^2+2bc^2+c^3} & \\
\qquad\qquad\qquad 0 &
\end{array}
$$

Exer. 31.

$$\begin{array}{rrrrr|rrr}
4 & 0 & -1 & 0 & 4 & 2 & 3 & 2 \\
4 & 6 & 4 & & & 2 & -3 & 2 \\
\hline
 & -6 & -5 & 0 & & & & \\
 & -6 & -9 & -6 & & & & \\
\hline
 & & 4 & 6 & 4 & & & \\
 & & 4 & 6 & 4 & & & \\
\hline
 & & & 0 & & & &
\end{array}$$

Exer. 32.

$$\begin{array}{rrrrrrrrr|rr}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -9 & 8 & 2 & -1 \\
2 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & & \\
\overline{2} & 4 & 6 & 8 & 10 & 12 & 14 & 16 & \overline{0} & & \\
 & \overline{3} & \overline{4} & \overline{5} & \overline{6} & \overline{7} & \overline{8} & \overline{0} & & &
\end{array}$$

Ans. $x+2x^2+3x^3+4x^4+5x^5+6x^6+7x^7+8x^8$.

MISCELLANEOUS INVESTIGATIONS.

(ALGEBRA, p. 49.)

Exer. 1.

This will be wrought by putting the proposed quantity under any of the forms, $\{(x^2+x)+1\}^2$, $\{x^2+(x+1)\}^2$, and $\{(x^2+1)+x\}^2$.

Exer. 2.

The proposed quantity may be put under the form

$$\{(x^3-x^2)+(x-1)\}^2, \text{ or } \{(x^3+x)-(x^2+1)\}^2,$$
$$\text{or } \{(x^3-1)-(x^2-x)\}^2, \text{or } \{(x^2+1)(x-1)\}^2.$$

Exer. 6.

The factors are the same as those in Exam. 9. in Multiplication, except that the signs of a are changed. To get the answer here, therefore, change the signs of a in the answer to that example.

Exer. 7.

The answer will be found by changing the signs of a in the answer to Exer. 9. in Division.

Exer. 10.

The quantity which is to be cubed may be written

$$(1-x)+x^2, \ (1+x^2)-x, \text{ or } 1-(x-x^2).$$

Exer. 12.

The factors may be written $(1+x^2)+x$ and $(1+x^2)-x$.

Exer. 15.

The proposed quantity is the same as $(x+a)^2-b^2$.

Exer. 22.

The dividend may be written $5(x^4-1)-x(x^2-1)$.

Exer. 23.

The first divisor may be written either $1+r^2$ or r^2+1, and the method of detached co-efficients may be employed.

The second divisor may be written either $1-2x+x^2$ or x^2-2x+1; and, by the method of detached co-efficients, or by the common method, the quotient will be found to be what is given in the ALGEBRA, or

$$\frac{1}{x^2}+\frac{2}{x^3}+\frac{3}{x^4}+\frac{4}{x^5}+\ldots\ldots+\frac{n}{x^{n+1}}+\frac{(n+1)x^{-n}-nx^{-n-1}}{x^2-2x+1}.$$

Exer. 24.

By a change of arrangement, and by § 51. &c., the given quantity may be put successively under the following forms:

$$(x^7+x^3)+(x^6+x^2)+(x^5+x)+(x^4+1);$$
$$(x^4+1)x^3+(x^4+1)x^2+(x^4+1)x+(x^4+1)1;$$
$$(x^4+1)(x^3+x^2+x+1);$$
$$(x^4+1)\{x^2(x+1)+1(x+1)\};$$
$$\text{and } (x^4+1)(x^2+1)(x+1).$$

Exer. 25.

As in the last, we shall have, successively,

$$(x^7-x^3)-(x^6-x^2)-(x^5-x)+(x^4-1);$$
$$(x^4-1)x^3-(x^4-1)x^2-(x^4-1)x+(x^4-1)1;$$
$$(x^4-1)(x^3-x^2-x+1);$$
$$(x^2+1)(x^2-1)\{x^2(x-1)-1(x-1)\};$$
$$(x^2+1)(x^2-1)(x^2-1)(x-1);$$
$$(x^2+1)(x+1)(x-1)(x+1)(x-1)(x-1);$$
$$\text{and } (x^2+1)(x+1)^2(x-1)^3.$$

Exer. 26.

$$\begin{array}{cccccccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\ldots\ldots & 1 & -1 \\
 & 1 & -1 & -1 & 0 & 1 & 1 & 0\ldots\ldots & & \\
 & \overline{1} & 1 & 0 & -1 & -1 & 0 & & & \\
 & & \overline{0} & -\overline{1} & -\overline{1} & \overline{0} & \overline{1} & & &
\end{array}$$

Hence $\dfrac{1}{1-r+r^2}=1+r-r^3-r^4+r^6+r^7-r^9-r^{10}+$ &c.

The second answer is found by changing the signs of all the odd powers of r; or it may be found by actual division.

Exer. 27.

Divide the first series by $1+r$, and the second by $1-r$, and call the quotients s_1 and s_2, &c.

Exer. 28.

By § 57., $(a+b)^2-c^2=(a+b+c)(a+b-c)$,
$(a+c)^2-b^2=(a+b+c)(a-b+c)$, and
$(b+c)^2-a^2=(a+b+c)(-a+b+c)$;
the sum of which is evidently $(a+b+c)(a+b+c)$ or $(a+b+c)^2$.

Exer. 29.

$$
\begin{array}{l|l}
x^2+2a^2 & x^2+a^2 \\
\underline{x^2+\ \ a^2} & \overline{1+a^2x^{-2}-a^4x^{-4}+a^6x^{-6}-\&c.} \\
\qquad a^2 & \\
\qquad \underline{a^2+a^4x^{-2}} & \\
\qquad\quad -a^4x^{-2} & \\
\qquad\quad \underline{-a^4x^{-2}-a^6x^{-4}} & \\
\qquad\qquad\qquad a^6x^{-4} & \\
\qquad\qquad\qquad \underline{a^6x^{-4}+a^8x^{-6}} & \\
\qquad\qquad\qquad\quad -a^8x^{-6},\ \&c. &
\end{array}
$$

By considering the terms of the quotient above obtained (the first term of which may be written a^0x^0), it will be seen that the indices of the powers of a are each, if increased by 2, double of the number expressing the *order* of the term in which it occurs; so that if n denote the order of the term, the index will be $2n-2$: and the indices of the powers of x are everywhere the same as those of the powers of a, except that they have contrary signs. Hence, the general term is $a^{2n-2}x^{-2n+2}$; and by taking $n=120$, we find the 120*th* term to be $a^{238}x^{-238}$.

Exer. 30.

By § 56., $(x+y)^3=x^3+3x^2y+3xy^2+y^3$, and $(x-y)^3=x^3-3x^2y+3xy^2-y^3$. The sum of these is $2x^3+6xy^2$, and their difference $6x^2y+2y^3$; which (§ 51.) are respectively equivalent to $2x(x^2+3y^2)$ and $2y(3x^2+y^2)$.

Exer. 31.

Find the actual product of the three factors, and it will be easily exhibited in the required form by means of § 51.

Exer. 32.

Find the actual powers, and proceed according to the directions in the question.

Exer. 33.

$(a^2+b^2)^2=a^4+2a^2b^2+b^4$, $(a^2-b^2)^2=a^4-2a^2b^2+b^4$, and $(2ab)^2=4a^2b^2$; the first of which is equal to the sum of the other two.

Exer. 34.

Proceed according to the method given above for solving Exer. 31.

FRACTIONS.

(Algebra, p. 66, 67, &c.)

In Exercises 18, 19, and 20. the common measures are readily found by inspection, by means of §§ 53, 57, 58, &c. They may also be found by means of § 81. or 82.

Exer. 21.

The fraction may be written $\dfrac{3(x^5+1)+2x(x^3+1)-x^2(x+1)}{(x^3+1)-5x(x+1)}$:
and (§ 60.) the numerator and denominator of this are both divisible by $x+1$.

Exer. 22.

The solution of this exercise is obtained at once from § 60., Algebra, page 46.

Exer. 23.

By the first method.

$$\begin{array}{l}
2 \quad 0 \quad -1 \quad -1)\,1 \quad 0 \quad -1 \quad -2 \quad 2 \\
\qquad\qquad\qquad\qquad\;\; 2 \\
\hline
\qquad\qquad\qquad\qquad\;\; 2 \quad 0 \quad -2 \quad -4 \quad 4(1 \\
\qquad\qquad\qquad\qquad\;\; 2 \quad 0 \quad -1 \quad -1 \\
\hline
2 \quad 0 \quad -1 \quad -1 \;\Big|\; -1 \quad -3 \quad 4 \\
2 \quad 6 \quad -8 \qquad\;\; \Big|\; -2 \quad 6 \\
\hline
\quad -6 \quad 7 \quad -1 \\
\quad -6 \; -18 \quad 24 \\
\hline
\qquad 25)25 - 25 \\
\hline
\qquad\quad 1 \; -1)\; -1 \quad -3 \quad 4(-1 \quad -4 \\
\qquad\qquad\qquad\;\; -1 \quad 1 \\
\hline
\qquad\qquad\qquad\qquad\;\; -4 \quad 4 \\
\qquad\qquad\qquad\qquad\;\; -4 \quad 4 \\
\hline
\qquad\qquad\qquad\qquad\qquad\;\; 0
\end{array}$$

Hence $x-1$ is the common measure; and by it divide both the numerator and denominator.

By the second method.

Here, (*c*) is got by subtracting the line before it from (*b*); (*d*) by taking the difference of the line before it and (*b*); and (*e*) by taking the difference of the two lines preceding it.

1	0	−1	−2	2	...(*a*)
2	0	−1	−1		......(*b*)
2	0	−2	−4	4	...(*a*) × 2
		1	3	−4	...(*c*)
		2	6	−8	...(*c*) × 2
	6	−7	1		......(*d*)
	6	18	−24		......(*c*) × 6
	25)	25	−25		......(*e*)
		1	−1, as before.		

Exer. 24.

By the first method.

3 20 −57 80 −50)	18	0	−5	44	−5(6
	18	120	−342	480	−300
		−120	337	−436	295

3	20	−57	80	−50	
40					
120	800	−2280	3200	−2000	−1
120	−337	436	−295		
	1137	−2716	3495	−2000	
	40				
	45480	−108640	139800	−80000	−379
	45480	−127723	165244	−111805	
	6361)	19083	−25444	31805	
		3	−4	5	

3 −4 5)	−120	337	−436	295(−40	59
	−120	160	−200		
		177	−236	295	
		177	−236	295	
			0		

Ans. Measure, $3x^2 - 4x + 5$.

By the second method.

$$
\begin{array}{rrrrrl}
3 & 20 & -57 & 80 & -50 & \ldots (a) \\
18 & 0 & -5 & 44 & -5 & \ldots (b) \\
18 & 120 & -342 & 480 & -300 & \ldots (a)\times 6 \\
\hline
 & 120 & -337 & 436 & -295 & \ldots (c) \\
\hline
180 & 0 & -50 & 440 & -50 & \ldots (b)\times 10 \\
\hline
177 & -20 & 7 & 360 & & \ldots\ldots\ldots (d) \\
\hline
7080 & -19883 & 25724 & -17405 & & \ldots (c)\times 59 \\
7080 & -800 & 280 & 14400 & & \ldots (d)\times 40 \\
\hline
6361) & 19083 & -25444 & 31805 & & \ldots (e) \\
\hline
 & 3 & -4 & 5, & & \text{as before.}
\end{array}
$$

In this operation, (c) is got by taking (b) from the line after it; and (d) by taking (a) from the line after (c). The line (e) is the difference of the two lines before it. The rest is plain.

Exer. 25.

The denominator is the product of the factors x^2 and $64x^3-27$; the former of which is not a divisor of the numerator. The latter (§ 58.) is the product of $4x-3$ and $16x^2+12x+9$; the former of which is found, by trial, to be a divisor of the numerator. The solution might also be obtained by either of the rules in §§ 81 and 82.

Exer. 26.

By the first method.

$$
\begin{array}{rrrrrrrrrr}
8 & -30 & 31 & 0 & -12)16 & 0 & -53 & 45 & 6(2 \\
 & & & & 16 & -60 & 62 & 0 & -24 \\
\hline
 & & & & 5)60 & -115 & 45 & 30 & \\
\hline
 & & & & 12 & -23 & 9 & 6 &
\end{array}
$$

8	−30	31	0	−12	12	−23	9	6		
3										
24	−90	93	0	−36	2					
24	−46	18	12							
	−44	75	−12	−36						
	6									
	−264	450	−72	−216(	−22					
	−264	506	−198	−132						
	−14)	−56	126	−84						
		4	−9	6)	12	−23	9	6(	3	1
					12	−27	18			
						4	−9	6		
						4	−9	6		
							0			

Ans. $4x^2-9x+6$.

By the second method.

Here the line before (c) is the sum of the two lines preceding it; and the one before (d) is the difference of the one before it and (a). The line before (e) is the difference between the line preceding it and (c); and the last line is the difference of the two immediately preceding ones.

16	0	−53	45	6	...(a)
8	−30	31	0	−12	...(b)
32	0	−106	90	12	...(a) × 2
5)40	−30	−75	90		
8	−6	−15	18		.......(c)
16	−60	62	0	−24	...(b) × 2
	5)60	−115	45	30	
	12	−23	9	6	...(d)
	36	−69	27	18	...(d) × 3
	7)28	−63	42		
	4	−9	6		.......(e)
24	−18	−45	54		.......(c) × 3
24	−46	18	12		.......(d) × 2
	28	−63	42, as before.		

Exer. 27.

By the first method.

By dividing the numerator and denominator by $2x$, and employing the resulting coefficients, we shall have the work as follows.

$$\begin{array}{l}
9\quad -9\quad -7\quad 15\quad -6)12\quad -11\quad -7\quad 12\quad -4 \\
\qquad\qquad\qquad\qquad\qquad\qquad 3 \\
\qquad\qquad\qquad\qquad\qquad\qquad \overline{36\quad -33\quad -21\quad 36\quad -12}(4 \\
\qquad\qquad\qquad\qquad\qquad\qquad 36\quad -36\quad -28\quad 60\quad -24 \\
9\quad -9\quad -7\quad 15\quad -6\ \ \big|\ \ 3\quad 7\quad -24\quad 12 \\
\underline{9\quad 21\quad -72\quad 36}\qquad \big|\ \ 3\quad -10 \\
\quad -30\quad 65\quad -21\quad -6 \\
\quad \underline{-30\quad -70\quad 240\quad -120} \\
\qquad 3)\underline{135\quad -261\quad 114} \\
\qquad\quad 45\quad -87\quad 38)3\quad 7\quad -24\quad 12 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{15} \\
\qquad\qquad\qquad\qquad\qquad\qquad 45\quad 105\quad -360\quad 180(1 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{45\quad -87\quad 38} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 192\quad -398\quad 180 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{15} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2880\quad -5970\quad 2700(64 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{2880\quad -5568\quad 2432} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -134)-402\quad 268 \\
\qquad\qquad\qquad\qquad\qquad\qquad 45\quad -87\quad 38\ \ \big|\ \ 3\quad -2 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{45\quad -30}\qquad\ \ \big|\ \ 15\quad -19 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad -57\quad 38 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-57\quad 38} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

Hence $3x-2$ is the common measure for the terms of the reduced fraction; and by multiplying this result by $2x$, we get $6x^2-4x$, the common measure for the terms of the given fraction. The required fraction is then found either by dividing the terms of the given one by the measure thus obtained; or, in a preferable way, by dividing the terms of

$$\frac{12x^4-11x^3-7x^2+12x-4}{9x^4-9x^3-7x^2+15x-6},$$

by $3x-2$, the measure first found.

By the second method.

In this operation, the line marked (c) is got by subtracting the line before it from the one immediately preceding; and (d) is the sum of the two lines before it. The line (e) is the difference between the line before it and line (c); and (f) is the difference of the line before it and (d). In the last place, (g) is the difference of the two lines immediately preceding it.

12	−11	−7	12	−4	. (a)
9	−9	−7	15	−6	. (b)
36	−33	−21	36	−12	. $(a)\times 3$
36	−36	−28	60	−24	. $(b)\times 4$
	3	7	−24	12	. (c)
18	−18	−14	30	−12	. $(b)\times 2$
18	−15	−7	6		. . . . . (d)
36	−30	−14	12		. . . . . $(d)\times 2$
33	−37	10			. . . . (e)
18	42	−144	72		. . . . . $(c)\times 6$
	57	−137	66		. . . . . (f)
	1089	−1221	330		. . $(e)\times 33$
	285	−685	330		. . $(f)\times 5$
	268)804	−536			. . . . (g)
	3	−2, as before.			

Exer. 28.

By the first method.

4	−15	8	3)1	−5	4	3	9	
			4					
			4	−20	16	12	36	(1
			4	−15	8	3		
				−5	8	9	36	
				4				
				−20	32	36	144	(−5
				−20	75	−40	−15	
				3	−43	76	159	

4	−15	8	3	−43 76 159
43				
172	−645	344	129	−4
172	−304	−636		
	−341	980	129	
	43			
	−14663	42140	5547	341
	−14663	25916	54219	
	16224)16224	−48672		
		1	−3.	

Measure, therefore, $x-3$.

By the second method.

Here (*c*) is obtained by taking the line before it from (*a*); and (*d*) by adding (*c*) to the line next after it; and (*e*) and (*f*) are respectively the difference and the sum of the two lines immediately preceding each of them.

	4	−15	8	3	. . (*a*)
1	−5	4	3	9	. . (*b*)
4	−20	16	12	36	. . (*b*) × 4
	5	−8	−9	−36	. . (*c*)
	48	−180	96	36	. . (*a*) × 12
	53	−188	87	. . . .	(*d*)
	20	−75	40	15	. . (*a*) × 5
	20	−32	−36	−144	. . (*c*) × 4
		43	−76	−159	. . (*e*)
	2809	−9964	4611		. (*d*) × 53
	1247	−2204	−4611		. (*e*) × 29
4056)	4056	−12168	. . . .		(*f*)
	1	−3, as before.			

Exer. 29.

By the first method.

	4	−12	5	14	−12)6	−11	9	−13	6	
					2					
					12	−22	18	−26	12(3	
					12	−36	15	42	−36	
4	−12	5	14	−12		14	3	−68	48	
7										
28	−84	35	98	−84	2					
28	6	−136	96							
	−90	171	2	−84						
	7									
	−630	1197	14	−588	−45					
	−630	−135	3060	−2160						
		2)1332	−3046	1572						
		666	−1523	786						

```
666 −1523   786)14       3      −68        48
                333
                4662     999   −22644    15984(7
                4662−10661       5502
                        11660  −28146     15984
                          333
                    3882780−9372618  5322672(5830
                    3882780−8879090  4582380
                    −246764)−493528   740292
```

The measure, therefore, is $2x-3$. 2 −3

By the second method.

In this operation, (c) is the difference of the two lines preceding it; and (d) is the sum of (a) and the line before (c). We find (e) by taking the sum of the line before it and (c); and (f) is the sum of the two lines preceding it. Lastly, (g) is the sum of the two lines immediately preceding it.

```
4     −12     5     14   −12 .. (a)
6     −11     9    −13     6 .. (b)
12    −36    15     42   −36 .. (a)×3
12    −22    18    −26    12 .. (b)×2
       14     3    −68    48 .. (c)
16    −34    23    −12 ...... (d)
64   −136    92    −48 ...... (d)×4
78   −133    24 ............. (e)
32    −68    46    −24 ...... (d)×2
32     10   −87 ............. (f)
     2262  −3857    696 ... (e)×29
      256     80   −696 ... (f)×8
1259)2518  −3777 .......... (g)
        2    −3, as before.
```

Exer. 30.

By the first method.

```
9  0  11  18  42   0−8)18  0  −5−18  33  0  −4(2
                       18  0  22  36  84  0−16
                     −3)−27−54−51   0  12
                          9  18  17  0  −4
```

9	0	11	18	42	0	−8	9 18 17 0 −4
9	18	17	0	4			1 −2
	−18	−6	18	46	0		
	−18	−36	−34	0	8		
		30	52	46	−8	−8	
		3					
		90	156	138	−24	−24	10
		90	180	170	0	−40	
		−8)	−24	−32	−24	16	
			3	4	3	−2	

3 4 3 −2)	9	18	17	0	−4	(3	2
	9	12	9	−6			
		6	8	6	−4		
		6	8	6	−4		
				0			

Measure, $3x^3+4x^2+3x-2$.

By the second method.

9	0	11	18	42	0	−8	(a)
18	0	−5	−18	33	0	−4	(b)
18	0	22	36	84	0	−16	(a)×2
		27	54	51	0	−12	(c)
36	0	−10	−36	66	0	−8	(b)×2
27	0	−21	−54	24			(d)
18)54	72	54	−36				(e)
3	4	3	−2				(f)
54	108	102	0	−24			(c)×2
27)81	108	81	−54				(g)
3	4	3	−2, as before.				

Here (c) is got by taking the difference of the two lines preceding it, and (d) by adding the line before it to (a). The line (e) is the difference of (c) and (d); and (g) is found by adding the line before it to (d).

Exer. 54.

By §§ 57. and 58. we have $\dfrac{(x+2a)(x-2a)(x-a)(x^2+ax+a^2)}{(x-a)(x+2a)}$ as the product; which, by contraction, becomes $(x-2a)(x^2+ax+a^2)$, or, by multiplication, $x^3-ax^2-a^2x-2a^3$.

Exer. 55.

Here the product is $\frac{a^4b^2}{b(x-a)(x-a)}$, or $\frac{a^4b}{(x-a)^2}$.

Exer. 57.

By reduction (§ 75.) the factors become $\frac{2y}{x+y}$ and $\frac{2x}{x-y}$; the product of which is $\frac{4xy}{x^2-y^2}$.

Exer. 65.

We have here (§ 75.) the quotient equal to $\frac{(a+1)a^2}{a(a^2-1)}$ or $\frac{a}{a-1}$.

Exer. 66.

Here the answer is $\frac{(x+b)x^2y}{x(x^2-b^2)}=\frac{xy}{x-b}$.

Exer. 67.

The quotient is $\frac{(ax^2+ab^2)(x^3-b^2x)}{(x^2-ax)(a^2x+a^3)}$; which, by dividing the numerator and denominator by a and x, and by actual multiplication becomes the answer.

Exer. 68.

By § 75. the quantities become $\frac{a^2}{x^2+a^2}$ and $\frac{x^2}{x^2-a^2}$; and the answer is found in the usual way.

Exer. 69.

By § 75. the given quantities become $\frac{x^4-a^4}{x^2-4a^2}$ and $\frac{x^2-a^2}{x-2a}$, or $\frac{(x^2+a^2)(x^2-a^2)}{(x+2a)(x-2a)}$ and $\frac{x^2-a^2}{x-2a}$; and the answer is found simply by dividing the numerator and denominator of the dividend by those of the divisor.

Exer. 70.

By § 75. the given quantities become $\frac{9x^4-28x^2+4}{x^2}$ and $\frac{3x^2-4x-2}{x}$: and the work is most readily completed by dividing the numerator and denominator of the first fraction respectively by those of the second.

Miscellaneous Exercises regarding Fractions.
(ALGEBRA, p. 74.)

Exer. 1.

By proceeding as the question directs, we get $a^2 \pm 2ab + b^2$, or (§ 53.) $(a \pm b)^2$.

Exer. 2.

The given fractions may be written $\frac{a^2(a-x)}{(a+x)(a^2-ax+x^2)}$ and $\frac{2(a+x)}{a-x}$; and the answer is obtained by indicating the multiplication, and omitting the factors common to the numerator and denominator.

Exer. 3.

Multiply the numerator and denominator of first fraction by $x-a$. Then perform the actual addition, and divide the numerator and denominator of the result by x^2+ax+a^2.

Exer. 4.

The numerators may be put under the forms,

$$(x+y)(x^2-xy+y^2) \text{ and } (x+y)^2\,;$$

and the denominators may be written

$$x(x^2+y^2)(x^2-y^2) \text{ and } x(x^2+y^2).$$

Then the answer is obtained by indicating the operation and omitting the factors, $x+y$, x, and x^2+y^2, in the numerator and denominator of the result.

Exer. 5.

By reducing the given fractions to equivalent ones having a common denominator, we obtain for numerators $x^2+2ax+a^2$ and $x^2-2ax+a^2$, and for denominator x^2-a^2. By taking half the sum of the squares of the numerators, we get $x^4+6a^2x^2+a^4$; and thence we derive the answer by dividing by $x^4-2a^2x^2+a^4$, the square of the denominator.

Exer. 6.

The sum of the first and third fractions is $\dfrac{2x}{x^2-a^2}$; and, after reducing this and the double of second fraction to the same denominator, we get the answer by taking the difference of the results.

Exer. 7.

The sums of the extremes and means are, respectively,

$$\frac{2x}{x^2-9a^2} \text{ and } \frac{2x}{x^2-a^2};$$

and the answer will be found by taking the latter of these from the former.

Exer. 8.

To find the answer, reduce the first and second fractions to the same denominator; square the results; from the sum of those squares take the square of the third fraction, and divide the numerator and denominator of the remainder by a^2-b^2.

Exer. 9.

This exercise is wrought without difficulty by reducing the first and second fractions to the same denominator, and taking the difference of the results; then by reducing the second and third in a similar manner, and taking their difference, &c.

Exer. 10.

To reduce the fractions by pairs to common denominators, multiply the terms of the first by $3x+1$, and those of the second by $x+1$; those of the second by $4x+1$, and of the third by $2x+1$, &c. Then, the answers will be obtained by taking the second of the results from the first, the fourth from the third, &c.

Exer. 11.

This presents no difficulty.

Exer. 12.

The numerators of the sum and difference of the given fractions are $2a^2+2$ and $4a$, and their common denominator a^2-1.

Exer. 13.

After making the changes indicated in the question, multiply the numerators and denominators of the first and second results by b, and those of the third by b^2.

Exer. 14.

The sum of the first and second fractions is $\frac{1+x^2}{1-x^2}$ and the difference of the third and fourth is the same. Hence, the quotient is 1.

Exer. 15.

The sum of the first and second fractions is $\frac{2}{1-x^2}$, and the difference of the others $\frac{2x}{1-x^2}$; and the answer is found by dividing the first result by the second, or simply the first numerator by the second.

Exer. 16.

Reject the common factors, $x-1$, x, and $x+1$, in the denominators, and then divide in the common way.

Exer. 17.

Multiply the numerator and denominator of the first fraction by $x+2$, and those of the second by $x-2$; then add in the usual way; and divide the numerator and denominator of the sum by x.

Exer. 18.

Proceed as in the last, only subtract instead of adding.

RADICALS OR SURDS.

(ALGEBRA, p. 80, &c.)

Exer. 1.

$\sqrt{75}=\sqrt{25}\times\sqrt{3}=5\sqrt{3}=5\times 1{\cdot}7320508=8{\cdot}660254.$

Exer. 2.

$\sqrt{117}=\sqrt{9}\times\sqrt{13}=3\sqrt{13}=$&c.

Exer. 3.

$\sqrt{1728}=\sqrt{576}\times\sqrt{3}=24\sqrt{3}=$&c.

Exer. 4.

$\sqrt{1500}=\sqrt{100}\times\sqrt{15}=10\sqrt{15}=$&c.

Exer. 5.

$\sqrt[3]{108}=\sqrt[3]{27}\times\sqrt[3]{4}=3\sqrt[3]{4}=$&c.

Exer. 6.

$\sqrt[3]{686}=\sqrt[3]{343}\times\sqrt[3]{2}=7\sqrt[3]{2}=$&c.

Exer. 7.

$$\sqrt{\frac{ab^2c^3}{d^4e^5}}=\sqrt{\frac{b^2c^2}{d^4e^4}}\times\sqrt{\frac{ac}{e}}=\frac{bc}{d^2e^2}\sqrt{\frac{ac}{e}};$$

$$\text{or }=\sqrt{\frac{ab^2c^3e}{d^4e^6}}=\sqrt{\frac{b^2c^2}{d^4e^6}}\times\sqrt{ace}=\frac{bc}{d^2e^3}\sqrt{ace}.$$

Exer. 8.

$$\sqrt[3]{\frac{a^4b-a^3b^2}{c^5}}=\sqrt[3]{\frac{a^3(ab-b^2)}{c^3c^2}}=\frac{a}{c}\sqrt[3]{\frac{ab-b^2}{c^2}};$$

$$\text{or}=\sqrt[3]{\frac{a^3bc(a-b)}{c^6}}=\frac{a}{c^2}\sqrt[3]{bc(a-b.)}$$

Exer. 9.

$$\sqrt{(3a^2b+6ab^2+3b^3)}$$
$$=\sqrt{(a^2+2ab+b^2)}\times\sqrt{3b}=(a+b)\sqrt{3b}.$$

Exer. 10.

$$\sqrt[3]{\frac{a^4-a^3}{b^4+b^5}}=\sqrt[3]{\frac{a^3}{b^3}}\times\sqrt[3]{\frac{a-1}{b+b^2}}=\frac{a}{b}\sqrt[3]{\frac{a-1}{b+b^2}};$$

$$\text{or}=\sqrt[3]{\frac{a^3}{b^3}}\times\sqrt[3]{\frac{(a-1)b^2(1+b)^2}{b^3(1+b)^3}}$$

$$=\frac{a}{b^2(1+b)}\sqrt[3]{(a-1)b^2(1+b)^2}.$$

Exer. 11.

$$\sqrt{\frac{ax^2-2ax+a}{x^3+2x^2+x}}=\sqrt{\frac{x^2-2x+1}{x^2+2x+1}}\times\sqrt{\frac{a}{x}}=\frac{x-1}{x+1}\sqrt{\frac{a}{x}};$$

$$\text{or}=\sqrt{\frac{x^2-2x+1}{x^2+2x+1}}\times\sqrt{\frac{ax}{x^2}}=\frac{x-1}{x(x+1)}\sqrt{ax}.$$

Exer. 12.

$$\sqrt{\frac{a^2b^2c^2-a^2c^4}{a^2b-2ab^2+b^3}}=\sqrt{\frac{a^2c^2}{a^2-2ab+b^2}}\times\sqrt{\frac{b^2-c^2}{b}}$$

$$=\frac{ac}{a-b}\sqrt{\frac{(b+c)(b-c)}{b}};$$

$$\text{or}=\sqrt{\frac{a^2c^2}{a^2-2ab+b^2}}\times\sqrt{\frac{b(b^2-c^2)}{b^2}}$$

$$=\frac{ac}{b(a-b)}\sqrt{b(b+c)(b-c)}.$$

Exer. 13.

$\sqrt{112}=\sqrt{16}\times\sqrt{7}=4\sqrt{7}$; $\sqrt{175}=\sqrt{25}\times\sqrt{7}=5\sqrt{7}$; and $\sqrt{343}=\sqrt{49}\times\sqrt{7}=7\sqrt{7}$; the sum of which is $16\sqrt{7}$.

Exer. 14.

$3\sqrt{44}=3\sqrt{4}\times\sqrt{11}=6\sqrt{11}$; and $\sqrt{275}=\sqrt{25}\times\sqrt{11}=5\sqrt{11}$: the difference of which is $\sqrt{11}$.

Exer. 15.

$\sqrt[3]{108}=\sqrt[3]{27}\times\sqrt{4}=3\sqrt[3]{4}$; and $\sqrt[3]{32}=\sqrt[3]{8}\times\sqrt[3]{4}=2\sqrt[3]{4}$. Then the answers are obtained by taking the sum and difference of these results,

Exer. 16.

$\sqrt{a^2x}=a\sqrt{x}$; $\sqrt{b^2x}=b\sqrt{x}$, and $\sqrt{c^2x}=c\sqrt{x}$. The answer is the sum of these.

Exer. 17.

$\sqrt{(x^3+2x^2y+xy^2)}=\sqrt{(x^2+2xy+y^2)}\times\sqrt{x}=(x+y)\sqrt{x}$. In like manner, $\sqrt{(x^3-2x^2y+xy^2)}=(x-y)\sqrt{x}$. The rest is easy.

Exer. 18.

$$3^{\frac{2}{3}}=3^{\frac{4}{6}}=(3^4)^{\frac{1}{6}}=81^{\frac{1}{6}}, \text{ and } 4^{\frac{1}{2}}=4^{\frac{3}{6}}=(4^3)^{\frac{1}{6}}=64^{\frac{1}{6}}.$$

Exer. 19.

The fractions equivalent to $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$, and having the same denominator, are $\frac{4}{8}$, $\frac{2}{8}$, and $\frac{1}{8}$; and therefore the given expressions become $2^{\frac{4}{8}}$ or $16^{\frac{1}{8}}$, $5^{\frac{2}{8}}$ or $25^{\frac{1}{8}}$, and $7^{\frac{1}{8}}$.

Exer. 20.

The indices are equivalent to $\frac{8}{12}$, $\frac{9}{12}$, and $\frac{10}{12}$. The rest is easy.

Exer. 21.

Here the indices are $\frac{12}{12}$, $\frac{6}{12}$, $\frac{4}{12}$, and $\frac{3}{12}$.

(ALGEBRA, p. 84.)

Exer. 22.

The given quantities, after a transformation of the first, are $2a^{\frac{2}{4}}$ and $3a^{\frac{1}{4}}$. Then the answer is $2\times 3a^{\frac{2}{4}+\frac{1}{4}}$, or $6a^{\frac{3}{4}}$.

Exer. 23.

The proposed quantities are equivalent to $x^{\frac{8}{12}}$ and $4x^{\frac{9}{12}}$, and the answer is readily found.

Exer. 24.

The product is $\sqrt{60a^2}$ or $2a\sqrt{15}$.

Exer. 25.

Take the product of the quantities in the vinculums, and prefix to it the sign of the square root.

Exer. 26.

Here (§ 57.) the product is $2-1$, or 1.

Exer. 27.

The given quantities are the same as $x^{\frac{6}{6}}$, $x^{\frac{3}{6}}$, and $x^{\frac{2}{6}}$; the product of which is $x^{\frac{11}{6}}$.

Exer. 28.

The product is $\sqrt[3]{168a^7}=\sqrt[3]{8a^6}\times\sqrt[3]{21a}=2a^2\sqrt[3]{(21a)}$.

Exer. 29.

Find the continual product of the quantities in the vinculums, and prefix to it the sign of the cube root.

Exer. 30.

To the product of the quantities within the vinculums prefix the sign of the square root.

Exer. 31.

The product of these (§ 57.) is $(x+2)-1$ or $x+1$.

Exer. 32.

To work this in the easiest manner, employ merely the coefficients 1 —2 3, and 1 4 6.

Exer. 34.

By dividing the first coefficient by the second, we get $\frac{3}{4}$; and by taking the second index from the first, we obtain $\frac{1}{2}$, which is the index of a in the answer.

Exer. 35.

This may be wrought either by multiplying the denominator of the first fraction by $3\sqrt{x}$; or by writing the first under the form $4x^{-\frac{1}{2}}$ and the second under the form $3x^{\frac{1}{2}}$, and then dividing the first coefficient by the second, and subtracting the second index from the first.

Exer. 36.

Multiply the first fraction by the reciprocal of the second.

Exer. 37.

Divide 30 by 5, and to the quotient prefix the sign of the square root.

Exer. 38.

Neglecting the radical signs, divide the first quantity by the second, and to the result prefix the sign $\sqrt[3]{}$.

Exer. 39.

Put the second quantity under the form $(x^2)^{\frac{1}{4}}$, and divide the first by it.

Exer. 40.

Multiply the first fraction by the reciprocal of the second.

Exer. 41.

The coefficients of the given quantities are 4, -9, 14, -19, and 4, and 1, -2, 3, and -4; and the division will be performed by means of them in the usual way.

Exer. 42.

As the indices of the first quantity are each $\frac{3}{8}$, the coefficients of that quantity are 1, 0, 0, and -1, and those of the second 1 and -1; and the quotient is readily obtained by means of these. It may be got also by means of § 60.

Exer. 43.

Multiply the first quantity by the reciprocal of the second, according to § 57., and simplify the product.

(Algebra, p. 87.)

Exer. 44. and 45.

In the first of these multiply the numerator and denominator of the quantity affected by the radical sign, by 11; and in the second by 11^2.

Exer. 46. and 47.

In the first of these multiply the numerator and denominator by $\sqrt{2}-1$; in the second multiply by $\sqrt{2}+1$.

Exer. 48.

Multiply the numerator and denominator by $\sqrt{2}+1$.

Exer. 49.

Multiply the numerator and denominator by $\sqrt{6}-\sqrt{5}$, and simplify the result.

Exer. 50.

Multiply the numerator and denominator of the given fraction by $\sqrt{(x+a)}+\sqrt{(x-a)}$, and simplify the result.

Exer. 51.

Multiply the numerator and denominator by $a\sqrt{x}-b\sqrt{y}$.

Exer. 52.

Multiply the numerator and denominator by $x+\sqrt{(x^2-a^2)}$, to get the first answer. In the second part the denominator is the same as

$$\sqrt{5}-\sqrt{10}+2\sqrt{5}-2\sqrt{10}+4\sqrt{5}, \text{ or } 7\sqrt{5}-3\sqrt{10};$$

and the answer will be found by multiplying the numerator and denominator by $7\sqrt{5}+3\sqrt{10}$.

Exer. 53.

The denominator may be written $\sqrt{5}-\sqrt{10}+2\sqrt{5}-2\sqrt{10}+4\sqrt{5}$, or $7\sqrt{5}-3\sqrt{10}$; and the answer will be obtained by multiplying the numerator and denominator by $7\sqrt{5}+3\sqrt{10}$.

Exer. 54.

Multiply the numerator and denominator by $\sqrt{(a+x)}+\sqrt{x}$, and divide the result by a.

Miscellaneous Exercises regarding Radicals.

(ALGEBRA, page 87.)

Exer. 55.

Multiply the numerator and denominator of the first fraction by $\sqrt{x}+1$, and those of the second by $\sqrt{x}-1$. In this way

the numerators become $x+2x^{\frac{1}{2}}+1$, and $x-2x^{\frac{1}{2}}+1$, and the common denominator is $x-1$. Then the answer is readily found by squaring the numerators thus obtained, by taking the sum and difference of the results, and by writing under them the square of the denominator, $x-1$.

Exer. 56.

In the quantities given in Exer. 55., write xy^{-1} for x, and multiply the numerators and denominators by $\sqrt{y}$; and, in the answers, change x^2 into x^2y^{-2} and x into xy^{-1}, and multiply the numerators and denominators of the results by y^2.

Exer. 57.

The numerator of the second fraction is the same as $(x^{\frac{1}{3}}+y^{\frac{1}{3}})(x^{\frac{1}{3}}-y^{\frac{1}{3}})$; and therefore, by rejecting the factor $x^{\frac{1}{3}}-y^{\frac{1}{3}}$, the numerator becomes $(x^{\frac{1}{3}}+y^{\frac{1}{3}})(x^{\frac{1}{3}}+y^{\frac{1}{3}})$, or $x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$. Then the answer is obtained by dividing this by the denominator, $x^{\frac{2}{3}}+y^{\frac{2}{3}}$.

Exer. 59.

Here the divisor is the same as $3x^{\frac{6}{12}}+0x^{\frac{7}{12}}+0x^{\frac{8}{12}}-x^{\frac{9}{12}}+2x^{\frac{10}{12}}$, and the dividend the same as $3x^{\frac{12}{12}}+0x^{\frac{13}{12}}+6x^{\frac{14}{12}}-x^{\frac{15}{12}}+11x^{\frac{16}{12}}-2x^{\frac{17}{12}}+4x^{\frac{18}{12}}-3x^{\frac{19}{12}}+6x^{\frac{20}{12}}$; and the work by means of the coefficients, will stand thus:

$$\begin{array}{ccccccccc|ccccc}
3 & 0 & 6 & -1 & 11 & -2 & 4 & -3 & 6 & 3 & 0 & 0 & -1 & 2 \\ \cline{10-14}
3 & 0 & 0 & -1 & 2 & & & & & 1 & 0 & 2 & 0 & 3 \\ \cline{1-7}
 & 0 & 6 & 0 & 9 & -2 & 4 & & & & & & & \\
 & & 6 & 0 & 0 & -2 & 4 & & & & & & & \\ \cline{3-9}
 & & & 0 & 9 & 0 & 0 & -3 & 6 & & & & & \\
 & & & & 9 & 0 & 0 & -3 & 6 & & & & & \\ \cline{5-9}
 & & & & & & 0 & & & & & & &
\end{array}$$

The answer, therefore, is $x^{\frac{6}{12}}+0x^{\frac{7}{12}}+2x^{\frac{8}{12}}+0x^{\frac{9}{12}}+3x^{\frac{10}{12}}$, or $x^{\frac{1}{2}}+2x^{\frac{2}{3}}+3x^{\frac{5}{6}}$.

Exer. 60.

Simply take the respective products of the given indices to find the indices of x in the answers.

Exer. 61.

By multiplying the numerators and denominators of the first and third fractions by $\sqrt{(1+x)}$, and those of the second and fourth by $\sqrt{(1-x)}$, we find the numerator of the dividend to be $2x$, and that of the divisor 2, each having the denominator $\sqrt{(1-x^2)}$. Then the answer is got by neglecting the common denominator, and dividing the numerator of the dividend by that of the divisor.

Exer. 62.

Divide the numerator and denominator by $a^{\frac{1}{2}}-x^{\frac{1}{2}}$, their common measure.

(ALGEBRA, p. 89, &c.)

Exer. 63.

By transposing $x-3$, and squaring the members of the resulting equation, we get $4x^2-3x-4=4x^2-4x+1$: and hence we readily find $x=5$.

Exer. 64.

To resolve this equation, transpose bx, square, &c.

Exer. 65.

Transpose $x+a$, square, &c.

Exer. 70.

Divide by $a+b-c$, and square the results.

Exer. 71.

By transposing $\sqrt{2x}$, squaring the members of the result, and rejecting $2x$, we get $-3a=9a-6\sqrt{2ax}$. Then, by transposing $-6\sqrt{2ax}$ and $-3a$, dividing by 6, squaring, &c., we get $x=2a$.

Exer. 72.

By multiplying both members of the equation by $\sqrt{a}$ and $\sqrt{x}-\sqrt{a}$, we get $\sqrt{ax}+\sqrt{ab}=\sqrt{bx}-\sqrt{ab}$; whence, by transposition, $\sqrt{ax}-\sqrt{bx}=-2\sqrt{ab}$: and from this the answer is obtained by dividing by $\sqrt{a}-\sqrt{b}$, and squaring the quotients.

Exer. 73.

By multiplying by $3b+\sqrt{x}$ and $b+\sqrt{x}$, we get,

$$4ab+4a\sqrt{x}+b\sqrt{x}+x=6ab+3b\sqrt{x}+2a\sqrt{x}+x.$$

Hence, rejecting x from each member, transposing $2a\sqrt{x}$, $3b\sqrt{x}$, and $4ab$, and dividing by 2, we get $a\sqrt{x}-b\sqrt{x}=ab$; and the answer is found by dividing by $a-b$ and squaring the quotients.

Exer. 74.

By squaring both members, we get

$$4a+x=4b+4x-4\sqrt{(bx+x^2)}+x.$$

Then, by rejecting x, dividing by 4, and transposing, we obtain $\sqrt{(bx+x^2)}=x-a+b$.* From this, by squaring both members, we get

$$bx+x^2=x^2-2ax+2bx+a^2+b^2-2ab;$$

and thence, by rejecting x^2, transposing, &c.,

$$2ax-bx=a^2-2ab+b^2=(a-b)^2 \text{ or } (b-a)^2;$$

and the value of x is found by dividing by $2a-b$.

Exer. 75.

Square both members and transpose 13. Then square both members of the resulting equation, and transpose 7. Lastly, square both members of the equation thus obtained, and transpose 3, &c.

* Or $\sqrt{(bx+x^2)}=x-(a-b)$. Hence by squaring both members, and rejecting x^2, we get

$$bx=-2(a-b)x+(a-b)^2;$$

whence the answer is readily obtained by transposing $-2(a-b)x$, or $-2ax+2bx$, &c.

Exer. 76.

By multiplying both members by $\sqrt{(x-a)}$ and $\sqrt{(x+a)}$, and contracting, we get $2x=b\sqrt{(x^2-a^2)}$; and by squaring, we obtain $4x^2=b^2x^2-a^2b^2$. Hence, by transposition, we have $b^2x^2-4x^2=a^2b^2$; and the answer is found by dividing by b^2-4, and extracting the square root.

Exer. 77.

By squaring both members, contracting, rejecting $2a^2$, and dividing by 2, we obtain $x^2-\sqrt{(x^4-a^2x^2+a^4)}=-b^2$. Hence, by transposition, $x^2+b^2=\sqrt{(x^4-a^2x^2+a^4)}$; and the answer is readily found, by squaring, &c., to be

$$x=\sqrt{\frac{b^4-a^4}{a^2-2b^2}}, \text{ or } x=\sqrt{\frac{a^4-b^4}{2b^2-a^2}}.$$

INVOLUTION, EVOLUTION, &c.

(ALGEBRA, p. 95.)

Exer. 5.

$$\begin{array}{rl}
 & x^4+4x^3-8x+4(x^2+2x-2 \\
 & x^4 \\
\hline
2x^2+2x\;) & 4x^3 \\
2x & 4x^3+4x^2 \\
\hline
2x^2+4x-2\;) & -4x^2-8x+4 \\
 & -4x^2-8x+4 \\
\hline
 & 0
\end{array}$$

Or thus:

$$\begin{array}{rrrrrrl}
 & 1 & 4 & 0 & -8 & 4\,(& 1 \quad 2 \quad -2 \\
 & 1 & & & & & \\
\hline
2 & 2\;) & 4 & 0 & & & \\
 & 2 & 4 & 4 & & & \\
\hline
2 & 4 & -2) & -4 & -8 & 4 & \\
 & & & -4 & -8 & 4 & \\
\hline
 & & & & 0 & &
\end{array}$$

Exer. 6.

$$
\begin{array}{llll}
 & x^6+8x^5-80x^3+128x+64\,(x^3+4x^2-8x-8 & & \\
 & x^6 & & \\
\hline
2x^3+4x^2 &)\ 8x^5 & & \\
4x^2 & \ 8x^5+16x^4 & & \\
\hline
2x^3+8x^2-8x &)\ -16x^4-80x^3 & & \\
-8x & \ -16x^4-64x^3+64x^2 & & \\
\hline
2x^3+8x^2-16x-8 &)\ -16x^3-64x^2+128x+64 & & \\
 & \ -16x^3-64x^2+128x+64 & & \\
\hline
 & \qquad\qquad 0 & &
\end{array}
$$

Or thus :

$$
\begin{array}{llllllllll}
 & 1 & 8 & 0 & -80 & 0 & 128 & 64\,(1 & 4\ -8\ -8 \\
 & 1 & & & & & & & \\
\hline
2 & 4) & 8 & 0 & & & & & \\
 & 4 & 8 & 16 & & & & & \\
\hline
2 & 8 & -8) & -16 & -80 & 0 & & & \\
 & & -8 & -16 & -64 & 64 & & & \\
\hline
2 & 8-16 & -8) & -16 & -64 & 128 & 64 & & \\
 & & & -16 & -64 & 128 & 64 & & \\
\hline
 & & & & 0 & & & &
\end{array}
$$

Exer. 7.

$$
\begin{array}{lll}
 & 1+x\,(1+\frac{x}{2}-\frac{x^2}{8}, \text{ \&c.} & \\
 & 1 & \\
\hline
2+\frac{x}{2} &)\ x & \\
\frac{x}{2} & \ x+\frac{x^2}{4} & \\
\hline
2+x-\frac{x^2}{8} &)\ -\frac{x^2}{4} & \\
-\frac{x^2}{8} & \ -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} & \\
\hline
2+x-\frac{x^2}{4} &)\qquad \frac{x^3}{8}-\frac{x^4}{64}, \text{ \&c.} &
\end{array}
$$

The root of $1-x$ will be the same, except that the odd powers of x will have the opposite signs.

By detached coefficients:

$$\begin{array}{llllllllll}
 & 1 & 1(1 & \frac{1}{2} & -\frac{1}{8} & +\frac{1}{16}, \&c. \\
 & 1 \\ \hline
2 & \frac{1}{2}) & 1 \\
 & \frac{1}{2} & 1 & \frac{1}{4} \\ \hline
2 & 1 & -\frac{1}{8}) & -\frac{1}{4} \\
 & & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{8} & \frac{1}{64} \\ \hline
2 & 1 & -\frac{1}{4} & \frac{1}{16}) & \frac{1}{8} & -\frac{1}{64} \\
 & & & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & -\frac{1}{64} & \frac{1}{256} \\ \hline
2 & 1 & -\frac{1}{4} & \frac{1}{8} &)-\frac{5}{64} & \frac{1}{64} & -\frac{1}{256}, \&c.
\end{array}$$

(ALGEBRA, p. 97.)

Exer. 8.

$$\begin{array}{ll}
 & 64x^6-288x^5+1080x^3-1458x-729(4x^2-6x-9 \\
 & 64x^6 \\ \hline
48x^4) & \quad -288x^5 \\ \hline
 & 64x^6-288x^5+432x^4-216x^3=(4x^2-6x)^3 \\ \hline
48x^4) & \qquad\qquad -432x^4 \\ \hline
\end{array}$$

We then find by cubing $4x^2-6x-9$, that it answers.

Exer. 9.

Here the fifth root of the first term is $3x$. Taking the fourth power of this, and multiplying it by 5, we get $405x^4$. Lastly, divide $-810ax^4$ by this, and the quotient is $-2a$, the remaining term of the root; and by actually raising $3x-2a$ to the fifth power, it is found to answer.

According to what is pointed out in the note, ALGEBRA, p. 97., the root would be found simply by taking the fifth roots of the first and last of the given terms.

(ALGEBRA, p. 100.)

Exer. 10.

$$\begin{array}{l}
2a+3b\sqrt{-1} \\
\underline{3a-2b\sqrt{-1}} \\
6a^2+9ab\sqrt{-1} \\
\underline{\qquad -4ab\sqrt{-1}+6b^2} \\
6a^2+5ab\sqrt{-1}+6b^2
\end{array}$$

Exer. 11.

$$\begin{array}{l}
a+b\sqrt{-1} \\
\underline{a+b\sqrt{-1}} \\
a^2+ab\sqrt{-1} \\
\underline{\qquad ab\sqrt{-1}-b^2} \\
a^2+2ab\sqrt{-1}-b^2=2\text{d power} \\
\underline{a+b\sqrt{-1}} \\
a^3+2a^2b\sqrt{-1}-ab^2 \\
\underline{\qquad\quad a^2b\sqrt{-1}-2ab^2-b^3\sqrt{-1}} \\
a^3+3a^2b\sqrt{-1}-3ab^2-b^3\sqrt{-1}.
\end{array}$$

Exer. 12.

$$\begin{array}{l}
1-2\sqrt{-1} \\
\underline{1-2\sqrt{-1}} \\
1-2\sqrt{-1} \\
\underline{\quad -2\sqrt{-1}-4} \\
-3-4\sqrt{-1}=\text{2d power.} \\
\underline{-3-4\sqrt{-1}} \\
9+12\sqrt{-1} \\
\underline{\quad +12\sqrt{-1}-16} \\
-7+24\sqrt{-1}=\text{4th power.}
\end{array}
\qquad
\begin{array}{l}
2-\sqrt{-1} \\
\underline{2-\sqrt{-1}} \\
4-2\sqrt{-1} \\
\underline{\quad -2\sqrt{-1}-1} \\
3-4\sqrt{-1}=\text{2d power.} \\
\underline{3-4\sqrt{-1}} \\
9-12\sqrt{-1} \\
\underline{\quad -12\sqrt{-1}-16} \\
-7-24\sqrt{-1}=\text{4th power.}
\end{array}$$

Exer. 13.

$$\begin{array}{l}
-\frac{1}{2}+\frac{1}{2}\sqrt{-3} \\
\underline{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}} \\
\frac{1}{4}-\frac{1}{4}\sqrt{-3} \\
\underline{\quad +\frac{1}{4}\sqrt{-3}-\frac{3}{4}} \\
-\frac{1}{2}-\frac{1}{2}\sqrt{-3}=\text{2d power.} \\
\underline{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}} \\
\frac{1}{4}+\frac{1}{4}\sqrt{-3} \\
\underline{\quad -\frac{1}{4}\sqrt{-3}+\frac{3}{4}} \\
1
\end{array}$$

In the second part the work will be the same, except the difference of a few signs.

Exer. 14.

In the first part, multiply the numerator and denominator each by $1+\sqrt{-1}$, and contract. In like manner, in the second multiply by $a+b\sqrt{-1}$.

Exer. 15.

25^2 is equal to $24^2-7^2\times-1$; and therefore (§ 57.) it is equal to $(24+7\sqrt{-1})(24-7\sqrt{-1})$. It is also equal to $7^2-24^2\times-1$, or (§ 57.) to $(7+24\sqrt{-1})(7-24\sqrt{-1})$. According to the second expression in the question, it is likewise equal to $15^2-20^2\times-1$, or to $20^2-15^2\times-1$; and hence the two remaining answers will be found by § 57.

Exer. 16.

By either of the methods given in §§ 81. and 82., we find the greatest common measure of the numerator and denominator to be $2x-a$. Then, if the numerator and denominator be divided by this, the fraction becomes $\dfrac{x+2a}{3x+a}$; the value of which, when $x=\frac{1}{2}a$, is 1.

Exer. 17.

Here, by either method, the common measure is found to be $x-3$; and by dividing the numerator and denominator by this, we get $\dfrac{x+1}{x+2}$; the value of which, when $x=3$, is $\dfrac{4}{5}$.

Exer. 18.

By means of the operation in the margin, which proceeds according to § 82., the greatest common measure is found to be $4x-3$. When the numerator and denominator are divided by this, they become respectively $3x^2-4x-3$ and $4x^2+3x-1$. Then, by taking $x=\frac{3}{4}$, the first becomes $-\frac{69}{16}$, and the second $\frac{7}{2}$ or $\frac{56}{16}$; and the answer is found by dividing the former by the latter.

12	−25	0	9	. . (*a*)
16	0	−13	3	. . (*b*)
48	0	−39	9	. . (*b*) × 3
36	25	−39		. . (*c*)
36	−75	0	27	. . (*a*) × 3
	100	−39	−27	. . (*d*)
900	625	−975		. . . . (*c*) × 25
900	−351	−243		. . . . (*d*) × 9
244)	976	−732		
	4	− 3		. . . . (*e*)

Exer. 19.

According to § 60, $x+a$ is a common measure of the numerator and denominator; and, dividing them successively by this, we obtain as quotients $x^4-ax^3+a^2x^2-a^3x+a^4$, and a^2-ax+a^2. Then, by substituting $-a$ for x, we get $5a^4$ and $3a^2$; and the answer is obtained by dividing the former of these by the latter.

Exer. 20.

The numerator and denominator may be put under the forms $(4x^3-4x^2)-(x^2-1)$, and $(4x^3-4x^2)+(4x^2-4x)-(x-1)$, each of which (§ 58.) is divisible by $x-1$. Dividing by this, therefore, we get $4x^2-(x+1)$, and $4x^2+4x-1$; the first of which becomes 2 and the second 7, by taking $x=1$.

The work by means of the coefficients, as in the margin, is very easy, and it gives $x-1$ as common measure, as before.

$$\begin{array}{rrrrl} 4 & -5 & 0 & 1 & \ldots (a) \\ 4 & 0 & -5 & 1 & \ldots (b) \\ \hline & 5) & 5 & -5 & \\ \hline & & 1 & -1 & \end{array}$$

Exer. 21.

In addition to what is stated in the note in p. 105. of the Algebra, it is easy to find (§ 60. 81. or 82.), that the greatest common measure of the numerator and denominator is $2x-3a$. Dividing both by this, we get $4x^2+6ax+9a^2$ and $2x+3a$, as the numerator and denominator of the equivalent fraction; the former of which becomes $27a^2$, and the latter $6a$, when $x=\frac{3}{2}a$; and the answer is found by dividing the former of these results by the latter.

Exer. 22.

Divide the numerator and denominator by x. Then, by taking $x=0$ in the results, the numerator becomes 1, and the denominator 0; and, therefore (§ 127.), the value of the fraction is infinite.

ARITHMETICAL AND GEOMETRICAL PROGRESSIONS.

(ALGEBRA, p. 108.)

Exer. 1.

Here $t_1=3$, and $d=2$; and therefore (§ 131.) $t_{10}=3+9\times 2=21$, and $t_{150}=3+149\times 2=301$.

Exer. 2.

Here $t_1=100$, and $d=-2$; then (§ 131.)
$t_{47}=100+46\times -2=8$, and $t_{60}=100+59\times -2=-18$.

Exer. 3.

In this exercise we have $d=3$, $t_1=4$, and $t_n=85$; and therefore (§ 131.) $n=\dfrac{85-4}{3}+1=28$.

Exer. 4.

Here we have evidently $t_1=1$, $t_{37}=100$, and $n=37$; and, therefore (§ 131.), $d=\dfrac{100-1}{36}=2\frac{3}{4}$.

Exer. 5.

The extremes in the first part are 1 and 1000, and the number of terms is 1000; and, therefore (§ 133.), the sum is $500\,(1+1000)=500{,}500$. In the second part we have $n=1000$, $t_1=1$, and $d=2$; and therefore (§ 133.)

$$s_{1000}=1000\times 1+500\times 999\times 2=1{,}000{,}000.$$

This latter answer, according to *Exam.* 6., would be found at once by squaring 1000.

Exer. 6.

Here $t_1=50$, $d=-1$, and $n=101$. Hence (§ 133.)

$$s_{101}=101\times 50+\tfrac{1}{2}\times 101\times 100\times -1=0.$$

Exer. 7.

Here $t_1=3$, and $d=2$; and, therefore (§ 133.),

$$s_n=n\times3+\tfrac{1}{2}n\ (n-1)\times2=3n+n^2-n=n^2+2n.$$

Exer. 8.

In this question, to find the number of steps, we have the extremes 1 and 884, and the number of terms 884. Hence (§ 133.) the required sum is half the product of 884 and 885, or the product of 442 and 885, that is 391,170. Now, by halving this we get 195,585, the entire number of feet ascended, since each step is supposed to be half a foot in height; and, dividing this by 5280, the feet in a mile, we get 37 miles, with 225 feet remaining, the whole height ascended or descended. Lastly, the breadth of each step being supposed to be 14 inches, the horizontal movement for each step in both ascent and descent will be the double of this, or $2\frac{1}{3}$ feet. To find the remaining answer, therefore, we multiply 391,170 by $2\frac{1}{3}$, and divide the product by 5280.

Exer. 9.

For the first 20 years, we have $t_1=£100$, $d=£100$, and $n=20$; and, therefore (§ 133.), $s_{20}=20\times100+10\times19\times100=21{,}000$, his gain during this period. For the second 20 years, we have $t_1=£200$, $d=£200$, and $n=20$; and, consequently, the sum of the series will be 42,000, the double of the last. Adding together, therefore, his original capital and the two gains thus found, we have for his whole fortune at the end of the forty years, £1000 + £21,000 + £42,000 = £64,000.

GEOMETRICAL PROGRESSION.

(ALGEBRA, p. 114.)

Exer. 1.

Here $t_1=1$ and $r=4$; and therefore, by (*a*) § 137., $s_6=\frac{4^6-1}{4-1}=\frac{4096-1}{3}=1365.$

Exer. 2.

We have here $t_1=3$ and $r=2$. Now $2^{16}=65536$; and by taking 1 from this, multiplying the remainder by $3(=t_1)$, and dividing the product by $1(=r-1)$, we get, according to § 137. (*a*), the answer.

Exer. 3.

Here $t_1=\frac{1}{27}$ and $r=3$; and therefore, by (*a*) § 137., $s_7=\frac{(3^7-1)\frac{1}{27}}{3-1}=40\frac{13}{27}$.

Exer. 4.

In this exercise we have $t_1=\frac{3}{2}$; and, by dividing the second term by the first, or the third by the second, we get $r=\frac{3}{2}$. Hence, by (*a*) § 137., to find the sum of eight terms, we raise $\frac{3}{2}$ to the eighth power, which is found to be $\frac{6561}{256}$. Then, taking $1(=\frac{256}{256})$ from this, dividing the remainder by $\frac{1}{2}(=\frac{3}{2}-1)$, and multiplying the result by $\frac{3}{2}(=t_1)$, we get the answer.

Exer. 5.

Here $t_1=\frac{2}{3}$, and $r=\frac{2}{3}$; and therefore, by (*b*) § 137., by raising $\frac{2}{3}$ to the eighth power, taking the result from 1, dividing the remainder by $\frac{1}{3}(=1-\frac{2}{3})$, and multiplying what is thus obtained by $\frac{2}{3}(=t_1)$, we get the answer to the first part. To get the answer to the second part, we divide, according to (*c*) § 138., $\frac{2}{3}(=t_1)$ by $\frac{1}{3}(=1-r)$.

Exer. 6.

In this exercise we have $t_1=\frac{11}{10}=1{\cdot}1$ and $r=\frac{11}{10}=1{\cdot}1$. Now, the sixth power of $1{\cdot}1$ is $1{\cdot}771561$. Taking 1 from this, and dividing the remainder by $0{\cdot}1(=1{\cdot}1-1)$, we get $7{\cdot}71561$, the product of which by $1{\cdot}1$ is the answer.

Exer. 7.

Here $t_1=\frac{10}{11}$ and $r=\frac{10}{11}$. Then, the sixth power of $\frac{10}{11}$ is $\frac{1000000}{1771561}$; and the first answer is found by taking this from 1, dividing the remainder by $\frac{1}{11}(=1-\frac{10}{11})$, and multiplying the result by the first term $\frac{10}{11}$. To find the second answer, we simply divide, according to (*c*) § 138., the first term $\frac{10}{11}$ by $\frac{1}{11}(=1-\frac{10}{11})$.

Exer. 8.

Here $t_1=\frac{1}{3}$; and $r=-\frac{2}{3}$, as is found by dividing any term by the one preceding it. Then, the fifth power of $-\frac{2}{3}$ being $-\frac{32}{243}$, we have, by (*b*) § 137., $s_5=\dfrac{(1+\frac{32}{243})\times\frac{1}{3}}{1+\frac{2}{3}}=\frac{55}{243}$, the first answer. The second, according to (*c*) § 138., is found by dividing $\frac{1}{3}$ by $\frac{5}{3}$.*

Exer. 9.

In this exercise we have $t_1=1$ and $r=-2=2\times(-1)$. Hence $r^n=2^n\times(-1)^n$, and $1-r=1+2=3$; and the answer is found by means of § 137. (*b*).

Exer. 10.

Here $t_1=\frac{1}{3}$ and $r=-\frac{1}{9}$. The fourth power of the latter is $\frac{1}{6561}$; and then, according to § 137. formula (*b*), we take this from 1, and we divide the remainder by $\frac{10}{9}=1-(-\frac{1}{9})$; and, lastly, we multiply the quotient by $\frac{1}{3}(=t_1)$, to find the first answer. To obtain the second, we divide, according to § 138. formula (*c*), $\frac{1}{3}$ by $\frac{10}{9}$.

Exer. 11.

In this question we have $t_1=1$ farthing, and $r=2$. Now, the twentieth power of 2 is readily found to be 1,048,576; and from this, by § 137. formula (*a*), we find the answer to be 1,048,575 farthings, which may be reduced in the usual way.

Exer. 12.

Here we have $r=3$, the twentieth power of which is found by multiplication to be 3,486,784,401. Then, according to formula (*a*) § 137., we subtract 1 from this, and divide the remainder by $2(=3-1)$, to find the answer in farthings.

* The sum of this series is evidently the difference of the sums of the two series,

$$\tfrac{1}{3},\ \tfrac{4}{27},\ \tfrac{16}{243},\ \text{\&c., and } \tfrac{2}{9},\ \tfrac{8}{81},\ \tfrac{32}{729},\ \text{\&c.,}$$

carried out to the proper number of terms; and hence we have another means of summing it and similar series, such as those in the next two exercises.

SIMULTANEOUS EQUATIONS.

(ALGEBRA, p. 121.)

Exer. 1.

First Method.

$$x+3y=17 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$
$$3x+y=11 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$
$$3x+9y=51 \ldots (1)\times 3 \ldots\ldots\ldots\ldots (3.)$$
$$8y=40 \ldots (3)-(2) \ldots\ldots\ldots (4.)$$
$$y=5 \ldots (4)\div 8 \ldots\ldots\ldots\ldots (5.)$$
$$x+15=17 \ldots (1) \text{ and } (5) \ldots\ldots\ldots (6.)$$
$$x=2, \text{ by transposition.}$$

Or thus:

$$9x+3y=33 \ldots (2)\times 3 \ldots\ldots\ldots\ldots (7.)$$
$$8x=16 \ldots (7)-(1) \ldots\ldots\ldots\ldots (8.)$$
$$x=2 \ldots (8)\div(8) \ldots\ldots\ldots\ldots (9.)$$
$$6+y=11 \ldots (2) \text{ and } (9)$$
$$y=5, \text{ by transposition.}$$

Second Method.

$$x=17-3y \ldots \text{from } (1) \ldots\ldots (10.)$$
$$x=\frac{11-y}{3} \ldots \text{from } (2) \ldots\ldots (11.)$$
$$\frac{11-y}{3}=17-3y \ldots (11) \text{ and } (10) \ldots (12.)$$
$$11-y=51-9y, \text{ \&c.}$$

Or thus:

$$y=\frac{17-x}{3} \ldots \text{from } (1) \ldots\ldots (13.)$$
$$y=11-3x \ldots \text{from } (2) \ldots\ldots (14.)$$
$$\frac{17-x}{3}=11-3x \ldots (13) \text{ and } (14) \ldots (15.)$$
$$17-x=33-9x, \text{ \&c.}$$

Third Method.

$x=17-3y$ . . . from (1) . . . . (16.)
$3x=51-9y$ . . . (16) $\times 3$ . . . . (17.)
$51-9y+y=11$ . . . . . . by (2) and (17) . . . . (18.)
&c. &c.

Or thus:

$y=11-3x$ . . . from (2) . . . . (19.)
$3y=33-9x$ . . . (19) $\times 3$ . . . . (20.)
$x+33-9x=17$ . . . . . . (1) and (20) . . . . (21.)
&c. &c.

Fourth Method.

$mx+3my=17m$ . . . . (1) $\times m$ . . . . (22.)
$(m+3)x+(3m+1)y=17m+11$. (2) and (22) . . . . (23.)
$3m+1=0$ . . . . (24.)
$m=-\frac{1}{3}$ . . . (from 24) . . . . (25.)
$(-\frac{1}{3}+3)x=17\times-\frac{1}{3}+11$ . . (from 23) . . (26.)
$8x=16$ . . . . (from 26) . . . . (27.)
$x=2$ . . . . . (from 27)
$m+3=0$, and $m=-3$.
$(-9+1)y=-51+11$ . . (from 23) . . . . . (28.)
$y=5$ . . . . . (from 28).

Otherwise:

From equations (1) and (2) we get by addition, and by dividing by 4, $x+y=7$; and the answers are easily got by subtracting the members of this equation from those of (2) and (1) successively.

Exer. 2.

First Method.

$x+3y=13$ . . . . (1.)
$3x-y=9$ . . . . (2.)
$3x+9y=39$ . . . . (1) $\times 3$ . . . . . . (3.)
$10y=30$ . . . . (3) $-$ (2)
$y=3$, by division . . . . (4.)
$x+9=13$, from (1) and (4).
$x=4$, by transposition.

Or thus :

$$9x-3y=27 \ldots (2)\times 3 \ldots\ldots (5.)$$
$$10x=40, \text{ by adding } (1) \text{ and } (5)$$
$$x=4, \text{ by division.}$$

Second Method.

$$x=13-3y \quad .. \text{ from } (1) \ldots (6.)$$
$$x=\frac{y+9}{3} \ldots \text{ from } (2) \ldots (7.)$$
$$\frac{y+9}{3}=13-3y \;..\; (6) \text{ and } (7) . \quad (8.)$$

Hence, by multiplying by 3, &c. we get $y=3$; and thence x is found from (6) or (7).

In following this method we might have found values of y, instead of finding those of x.

Third Method.

By trebling the members of (6), we get $3x=39-9y$; and by substituting this in (2), we obtain $39-9y-y=9$; whence y is easily found.

Fourth Method.

$$mx+3my=13m \ldots (1)\times m \ldots\ldots (9.)$$
$$(m+3)x+(3m-1)y=13m+9 \;..\; (9)+(2) \ldots (10.)$$
$$3m-1=0, \text{ and } m=\tfrac{1}{3} \ldots\ldots (11.)$$
$$(\tfrac{1}{3}+3)x=\tfrac{13}{3}+9, \text{ from } (10) \text{ and } (11) . (12.)$$

Hence x is found by multiplying by 3, &c.; and y may be found by taking $m+3=0$ in (10).*

Exer. 3.

First Method.

$$x+\tfrac{1}{2}y=16 \ldots\ldots (1.)$$
$$\tfrac{1}{2}x-\tfrac{1}{3}y=1 \ldots\ldots (2.)$$
$$2x+y=32 \ldots (1)\times 2 \ldots\ldots (3.)$$
$$3x-2y=6 \ldots (2)\times 6 \ldots\ldots (4.)$$

* The solution may also be readily obtained by adding half the sum of equations (1) and (2) to equation (2).

Then, to find x, add the members of equation (4) to twice those of equation (3), and divide the result by 7. To find y, take the doubles of the members of equation (4) from the trebles of those of equation (3), and divide the remainder by 7.

Second Method.

From equations (3) and (4) we have $x = \dfrac{32-y}{2}$, and $x = \dfrac{6+2y}{3}$, and the value of y is found by equalling these. Then x may be found from either of the preceding expressions: or from equations (3) and (4) we get $y=32-2x$, and $y=\dfrac{3x-6}{2}$; and x will be found from the equation obtained by equalling these.

Third Method.

From (3) we get $y=32-2x$, and therefore $2y=64-4x$; by substituting which in (4), we obtain $3x-64+4x=6$; and thence we get $x=10$.

Fourth Method.

Multiply the members of (1) by m, and add the result to those of (2): then $(m+\frac{1}{2})x+(\frac{1}{2}m-\frac{1}{3})y=16m+1$. By taking $\frac{1}{2}m-\frac{1}{3}=0$, and consequently $m=\frac{2}{3}$, this equation becomes $(\frac{2}{3}+\frac{1}{2})x=\frac{32}{3}+1$; whence $x=10$; while, by taking $m+\frac{1}{2}=0$, and therefore $m=-\frac{1}{2}$, the same equation becomes $(-\frac{1}{4}-\frac{1}{3})y=-8+1$, and gives $y=12$.

Exer. 4.

First Method.

$$\tfrac{3}{4}x-\tfrac{1}{3}y=1 \ldots\ldots (1.)$$
$$\tfrac{2}{3}x+\tfrac{3}{4}y=26 \ldots\ldots (2.)$$

By multiplying each of these by 12, we get

$$9x-4y=12 \ldots\ldots (3.)$$
$$8x+9y=312 \ldots\ldots (4.)$$

Multiply the members of (3) by 8, and those of (4) by 9: then

$$72x-32y=96 \ldots\ldots (5.)$$
$$72x+81y=2808 \ldots\ldots (6.)$$

Take the upper of these from the lower, and divide by 113: then $y=24$. To find x, substitute 24 for y in (3); transpose and divide by 9. We may also find y by multiplying the members of (3) by 9, and those of (4) by 4, adding the results, &c.

Second Method.

From (3) and (4) we get $x=\dfrac{12+4y}{9}$ and $x=\dfrac{312-9y}{8}$; and y will be found from the equation obtained by equalling these values of x. In like manner from (3) and (4) we should have $y=\dfrac{9x-12}{4}$, and $y=\dfrac{312-8x}{9}$; and x would be found by equalling these values of y.

Third Method.

From (3) we have $x=\dfrac{12+4y}{9}$; and therefore $8x=\dfrac{96+32y}{9}$. Substitute this in (4); then $\dfrac{96+32y}{9}+9y=312$, &c.

Exer. 5.

Third Method.

From the first equation $x=27-y$, and therefore $2x=54-2y$. By substituting this value of $2x$ in the second equation, we get $\dfrac{54-2y-y}{3}-\frac{1}{2}y=6$, or $18-y-\frac{1}{2}y=6$; whence y is easily found.

Exer. 6.

From the first equation, by multiplying by 3 and dividing by 2, we get $2x+y=33$; while, by multiplying by 5 and dividing by 3, we get from the second $x+y=20$. Take the members of this from those of the one formerly obtained; then $x=13$. The rest is easy.

Exer. 7.

By freeing the equations of fractions, &c. we get $4x-3y=4$, and $3x+y=3$. Multiply the members of the latter by 3, and

add the results to the former: then $13x=13$; whence $x=1$; and, by substituting this in either of the former, we get $y=0$.
The same would be obtained by multiplying the first of the equations found above by 3, and the second by 4, and taking the first result from the second; as we should thus get $13y=0$, and therefore $y=0$.

Exer. 8.

By multiplying the first equation by 6, we get $3x-3-(2y+2)=6$, and thence $3x-2y=11$. From the second equation, in like manner, by multiplying by 12, transposing and contracting, we obtain $4x+3y=43$. Then, to employ the fourth method, we may multiply the members of the former of these equations by m, and by adding the members of the second to the results, we get

$$(3m+4)x-(2m-3)y=11m+43.$$

This, by taking $2m-3=0$, and consequently $m=\frac{3}{2}$, becomes $\left(\frac{9}{2}+4\right)x=\frac{33}{2}+43$; whence, by multiplying by 2, &c. we readily find $x=7$.

The same equation, again, by taking $3m+4=0$, and consequently $m=-\frac{4}{3}$, becomes $-\left(-\frac{8}{3}-3\right)y=-\frac{44}{3}+43$, or by multiplying by 3, $(8+9)y=129-44$; whence $y=5$.

Exer. 9.

By clearing the second equation of fractions, transposing, &c., we get $11y=7x$; whence, by dividing by 11 and multiplying by 5, we obtain $5y=\frac{35x}{11}$. Then, by substituting this in the first equation, and by easy operations, we find $x=11$.

Exer. 10.

Here, by multiplying the members of the first equation by 3, we get $3x^{-1}+3y^{-1}=15$; and adding these equals to the members of the second equation $(5x^{-1}-3y^{-1}=1)$, we get $8x^{-1}=16$; and therefore $x^{-1}=2$, and $x=\frac{1}{2}$. Then, by substituting 2 for x^{-1} in the first equation, we find $y^{-1}=3$, and consequently $y=\frac{1}{3}$

Exer. 11.

By freeing the given equations of fractions, &c., we obtain

$$7x+2y=30, \text{ and } 3x+10y=38.$$

Then, by taking the members of the second from 5 times those of the first, we get $32x=112$; whence $x=3\frac{1}{2}$; and the first of the equations found above gives $y=\dfrac{30-7x}{2}=2\frac{3}{4}$.

Exer. 12.

By clearing the given equations of fractions, &c., we get $2x-3y=-5$, and $x+3y=20$. Hence, by addition, we have $3x=15$, and therefore $x=5$. The finding of y presents no difficulty.

Exer. 13.

By multiplying the first equation by 3, and the second by 4, and by transposing, &c., we obtain $3x-2y=-6$, and $-3x+4y=24$; whence, by adding, we have $2y=18$, and therefore $y=9$. Having thus found y, we can readily find x from either of the foregoing equations, or from either of the given ones, particularly the first.

Exer. 14.

By multiplying the members of the first of the given equations by 2 and 5, and those of the second by 4; by transposing, &c., we get

$$x+7y=40, \text{ and } -3x-3y=-24;$$

the latter of which, by dividing by -3, becomes $x+y=8$. By taking the members of this last equation from those of $x+7y=40$, we obtain $6y=32$; and therefore $y=5\frac{1}{3}$. Lastly, by taking this from 8, we get $x=2\frac{2}{3}$.

Exer. 15.

Multiply the members of the second equation by a, and from the results subtract the members of the first, and there will remain $(a^2-1)x=ac-b$; and the value of x is then found by dividing by a^2-1. In like manner, to find y, multiply the

members of the first by a, from the results take the members of the second, and divide by a^2-1.

The solution may also be readily obtained by substituting 1 for a_1 and for b_2, a for b_1, and for a_2, b for c_1, and c for c_2, in the values found for x and y in the ALGEBRA, p. 117., and changing the signs in the numerators and denominators.

Exer. 16.

By clearing the given equations of fractions, transposing, &c., we get $4x+3y=100$, and $7x+4y=150$. Hence the solution may be effected by any of the usual methods.

Exer. 17.

By transposing -1 in each of the given equations, the second members will become $x+1$ and $y+1$; and then by squaring, transposing, and dividing the first by -2, we get $x-y=-1$, and $3x-2y=1$: whence the values of x and y may be readily found by any of the common methods.

Exer. 18.

The second equation is the same as $ax+a'x+by+b'y=c+c'$; and by taking from the members of this those of the first equation, we obtain $a'x+b'y=c'$. Then, from this and from the first equation, the values of x and y will be found by any of the usual means. They may also be found very easily by means of the formulas (6) and (10) in the ALGEBRA, p. 117.

Exer. 19.

By multiplying by b in the first equation, and by b' in the second, we obtain

$$abx+y=bc, \text{ and}$$
$$a'b'x+y=b'c'.$$

Taking the latter from the former we get

$$(ab-a'b')x=bc-b'c';$$

and the value of x is found by dividing by $ab-a'b'$, The value of y is easily obtained by multiplying the first of the equations found above by $a'b'$, and the second by ab: then, by subtraction,

we get $(ab-a'b')y=abb'c'-a'bb'c$; and the required value is got by dividing by $ab-a'b'$.

The answers might also be obtained from the values of x and y found in Exam. 2. p. 117.

ALGEBRA, p. 124.

Exer. 20.

By doubling and trebling the members of the first equation, and taking from the first results the members of the second equation, and from the others those of the third, we get $7y+2z=49$, and $2y+14z=108$, or $y+7z=54$. From these two equations, the values of y and z will be found by any of the usual methods; and the value of x will be found from the first of the given equations by transposition.

The solution may also be easily effected by the method of substitution; as, by the first equation, we have $x=34-2y-3z$. Then, by doubling and trebling these equals, and by substituting the second member of the first result for $2x$ in the second equation, and the second member of the other for $3x$ in the third, two equations will be obtained containing only y and z.

Exer. 21.

By doubling the members of the first equation, and adding those of the second equation to the results, we get $13x-5y=120$. Again, by trebling the members of the first equation, and doubling those of the third, and taking the second results from the first, we obtain $11x-2y=106$. We have thus got two equations which contain only x and y, and which may therefore be resolved in any of the usual ways.

Exer. 22.

This exercise may be easily solved in different ways. Thus, to eliminate z, we get from the first and third equations, by addition, $5x+4y=13$; and, from the second and twice the first, we obtain by subtraction, $x-3y=-5$; whence x and y may be found.

If we wish to eliminate y, we may add the first and second equations; and also the third and three times the second.

Lastly, x may be eliminated by taking the second equation from the treble of the first, and the fourth from its quadruple.

By the method of substitution an expression may be found for either x, y, or z from the first equation and substituted in the others.

By the method of comparison, values of x, y, or z (particularly of y or z) may be easily found from the three given equations. Then by equalling two of these values, and also by equalling one of these two to the remaining one, two equations will be found containing only two unknown quantities.

The solution may also be readily effected by the fourth method (ALGEBRA, §§ 145. and 147.), though perhaps not quite so easily as by the other methods.

Exer. 23.

Out of the various ways in which this question may be solved, the following is very simple. Subtract the double of the second equation from the first, and there will result

$$\tfrac{1}{2}y - z = -8 + 2y - \tfrac{1}{3}z;$$

whence by transposition, &c., $\frac{3}{2}y + \frac{2}{3}z = 8$. Again, by taking twice the third equation from the second, we get $z = 23 - 5y$: and from these two equations the values of y and z are easily found.

ALGEBRA, p. 126.

Exer. 24.

By dividing the members of the second equation by those of the first, there is obtained (ALGEBRA, § 57.) $x - y = \frac{b}{a}$. Then, from this equation and the first of the given ones, the values of x and y are obtained by means of § 52., ALGEBRA, p. 44.

Exer. 25.

The two equations give, by subtraction, $2x - 4y = 2$; and from this and the first of the given equations, we get by another

subtraction, $x=3$. Lastly, from the equation found above, we have $y=\frac{1}{2}(x-1)=1$.

We should also get x at once by doubling the members of the first equation, and taking from the results those of the second.

Exer. 26.

From the three given equations we obtain, by addition, and by dividing by 2,

$$x^{-1}+y^{-1}+z^{-1}=s,$$

where $s=\frac{1}{2}(a+b+c)$. Then, by subtracting the third, the second, and the first of the given equations successively from the result just obtained, and taking the reciprocals of the remainders, we get the values of x, y, and z.

Exer. 27.

Take the first equation from the second, and the second from the third, and there will remain

$$y+2z=11, \text{ and } y+3z=15.$$

Then, by taking the first of these results from the second, we get $z=4$; and from the first of the same results, we have $y=11-2z=3$. Lastly, from the first of the given equations, we have $x=9-y-z=2$.

The question might also be solved by taking the double of the second equation from the sum of the first and third, as z would thus be obtained at once. Then y would be found by taking the first equation from the second, substituting for z its value, &c.

Exer. 28.

The continual product of the members of the three equations is $x^4y^4z^4=abc$: whence $xyz=a^{\frac{1}{4}}b^{\frac{1}{4}}c^{\frac{1}{4}}$. Then, if the members of the three given equations be successively divided by the equals just found, the quotients will be the values of x, y, and z.

Exer. 29.

From the given equations we obtain, by addition, and by dividing by 2, $x+y+z+v=s$. Then the answers are found by taking, successively, from these equals the given equations, and halving the remainders.

ALGEBRA, p. 132.

Exer. 30.

Here, according to the notation in the question, we have

$$x = my, \text{ and } x + a = n(y + b) = ny + nb.$$

Then, by substituting my for x in the second of these, by transposing ny and a, and by dividing by $m-n$, we find

$$y = \frac{nb-a}{m-n}; \text{ and, consequently, } x\ (=my) = \frac{m(nb-a)}{m-n}.$$

Now, these values of x and y are in their lowest terms; and as they become vanishing fractions, when $m=n$, and $nb=a$, the question (Alg., § 129.) is indeterminate. In that case, in fact, there is but one independent equation, the members of the second becoming identical, since $x=my$ or ny, and $a=nb$. It will be seen, too, that in the same case, the quantities of rum and water added are exactly proportional to the original quantities.

Again, in the case in which $m=n$, but nb not equal to a, we shall have the numerators of the values of x and y finite quantities, and the denominators each $=0$; and therefore (ALG., § 127.) the values of x and y are infinite. In this case, therefore, the question is impossible.

When x and y come out negative, the second of the original equations becomes $-x+a=-ny+nb$, or, by changing the signs, $x-a=ny-nb$; which shows, that x and y being the original quantities, a and b are subtracted.

This question and its solution will be well illustrated by means of numbers. Thus by taking $m=10$, $a=4$, $b=1$, and $n=9$, we should have $x=50$, and $y=5$. Again, if $m=n=10$, $a=20$, and $b=2$, x might be taken of any value, and y one tenth of it. Thus $x=30$, and $y=3$, would be found to answer; as would also $x=50$ and $y=5$.

If, however, $m=n=10$, $a=20$, and $b=3$, we should have $y=\frac{10}{0}=\infty$, and $x=\frac{100}{0}=\infty$; so that no finite values of x and y will answer to these data. The greater x and y are, however, the more nearly will they satisfy the question. Thus, if $x=1000$ and $y=100$, the additions of a and b will give 1020 and 103; the tenth part of the first of which differs from the second by 1; while if $x=1,000,000$ and $y=100,000$, the same additions will

give 1,000,020 and 100,003, in which also the tenth of the former differs from the latter bv 1. In the first case, therefore, there is an error of 1 in 102, while, in the latter, there is only the same error in 100,002.

The case in which x and y come out negative, will be illustrated by taking $m=10$, $n=9$, $a=20$, and $b=1$; as x would be found to be -110 and $y=-11$. Now these will be answers to the question in which the original quantities of rum and water were 110 and 11 gallons, and in which the former is *diminished* by 20 gallons and the latter by 1; as the one remainder, 90, is nine times the other, 10.

Exer. 31.

Let $\frac{x}{y}$ be the required fraction. Then, by the question, $\frac{x}{y+1}=\frac{1}{2}$, and $\frac{x+1}{y}=\frac{3}{5}$: and the values of x and y are easily found by clearing these equations of fractions, and by resolving the equations so obtained.

The following *general solution* will comprehend that of this particular question, as well as those of all similar questions. Assume

$$\frac{x+a}{y+b}=c, \text{ and } \frac{x+a_2}{y+b_2}=c_2.$$

Then, by multiplying by the denominators, we get

$$x+a=cy+bc \ldots\ldots\ldots\ldots(1)$$
$$\text{and } x+a_2=c_2y+b_2c_2 \ldots\ldots(2).$$

To eliminate x, subtract the latter from the former: then

$$a-a_2=(c-c_2)y+bc-b_2c_2.$$

Hence, by taking the second member first, transposing $bc-b_2c_2$, and dividing by $c-c_2$, we get

$$y=\frac{a-a_2-(bc-b_2c_2)}{c-c_2}.$$

The value of x may be obtained by substituting in either of the equations (1) and (2), the value just found for y, and resolving the result for x: or it may be found, perhaps more easily, by multiplying the members of equation (1) by c_2, and those of equation (2) by c, by subtracting the first results from the second, and by

resolving the equation so obtained for x. By this means we get

$$x=\frac{ac_2-a_2c-(b-b_2)cc_2}{c-c_2}.\,*$$

In the present exercise we have $a=0$, $a_2=1$, $b=1$, $b_2=0$, $c=\frac{1}{2}$, and $c_2=\frac{3}{5}$; and, by substituting these in the general values of x and y, we find $x=8$, and $y=15$.

Exer. 32.

Here we have $\frac{x+1}{y+1}=\frac{1}{2}$, and $\frac{x-1}{y-1}=\frac{1}{3}$; and the values of x and y are easily obtained by clearing the equations of fractions, &c.

The solution will be readily obtained from the general values of x and y found in the last exercise by taking $a=1, b=1$, $a_2=-1$, $b_2=-1$, $c=\frac{1}{2}$, and $c_2=\frac{1}{3}$.

Exer. 33.

Here, putting x to denote the money of the first person, and y that of the second, we have

$$x+\frac{1}{2}y=100, \text{ and } \frac{1}{3}x+y=100;$$

the resolution of which equations presents no difficulty.

The solution might also be effected by the assumption of only one unknown quantity, but scarcely so simply.

A general solution would be had by assuming $\frac{1}{2}=a$, $\frac{1}{3}=b$, the first amount (here £100) $=c$, and the second (here also £100) $=c_2$: and we should thus get

$$x=\frac{c-ac_2}{1-ab}, \text{ and } y=\frac{c_2-bc}{1-ab}.$$

* It will be of some interest to consider and interpret the values of x and y, which will be obtained, when the given numbers have some peculiar relations, such as when $c=c_2$, &c.

Exer. 34.

If x be the common difference, the three numbers will be represented by $a-x$, a, and $a+x$: and, by the question,

$$(a-x)\,(a+x), \text{ or } a^2-x^2=b:$$

whence $x=\sqrt{(a^2-b)}$; and the extremes are found at once.

It is plain, that if b, the product of the extremes, be greater than a^2, the square of the mean, the radical will be imaginary, and the question absurd. It appears also, that the square of the mean is equal to the product of the extremes together with the square of the common difference.

Exer. 35.

Denoting the numbers by $a-x$, a, and $a+x$, we have, in the first part of the exercise,

$$(a-x)^2+(a+x)^2=b, \text{ or } 2a^2+2x^2=b;$$

whence, by transposition, by division by 2, &c., we obtain $x=\sqrt{(\frac{1}{2}b-a^2)}$; and we find the extremes by taking the difference and sum of this and the mean. In this part of the exercise, it is plain that a^2 must not exceed $\frac{1}{2}b$.

In solving the second part, we have

$$(a+x)^2-(a-x)^2=b, \text{ or } 4ax=b;$$

whence x is easily found, and thence the extremes.

In this part, it is plain that the results can never be imaginary. If, however, $\frac{b}{4a}$ be greater than a, the first extreme would be negative; and therefore the data would not be such as to answer a question of the ordinary arithmetical kind. We should have an instance of this, if $a=2$ and $b=40$; as the extremes would be found to be -3 and 7, and consequently the three numbers would be -3, 2, and 7.

Exer. 36.

In questions regarding an *even* number of quantities in arithmetical progression, it is often advantageous to assume characters to denote half the common difference, and half the sum of the extremes (or of any two terms equally distant from them). In this way, we shall have in the present exercise $x-3y$, $x-y$, $x+y$, and $x+3y$, to represent the numbers. Then, by the question,

$$(x-3y)^2+(x+3y)^2=a, \text{ or } 2x^2+18y^2=a, \text{ and}$$
$$(x-y)^2+(x+y)^2=b, \text{ or } 2x^2+2y^2=b.$$

From these two equations we get, by subtraction, $16y^2=a-b$, and thence $y=\frac{1}{4}\sqrt{(a-b)}$. Again, by taking the first equation from 9 times the second, we obtain $16x^2=9b-a$, and thence $x=\frac{1}{4}\sqrt{(9b-a)}$; and from these values of x and y, the answers are easily found. From considering the values of y and x, it will be seen, that a cannot be less than b nor greater than $9b$.

Exer. 37.

Adopting the same notation as in the last exercise, we have

$$(x+3y)^2-(x-3y)^2=a, \text{ or } 12xy=a, \text{ and}$$
$$(x+y)^2-(x-y)^2=b, \text{ or } 4xy=b.$$

Now multiplying the latter of these by 3, we get $12xy=3b$; and therefore a is necessarily equal to $3b$, whatever may be the values of x and y. We have thus, therefore, in reality, only one equation for determining x and y, so that they may have any values whatever, provided 12 times their product be equal to a, or 4 times the same equal to b.

As an example, suppose $b=144$, and $a=432$, then the fourth of b being 36, we may take $x=36$ and $y=1$, and the numbers will be 33, 35, 37, and 39; which will be found to answer, as the difference of the squares of 33 and 39 is 432, while the difference of the squares of 35 and 37 is 144. We might take also $x=18$, and $y=2$; and we should find the terms to be 12, 16, 20, 24, which would answer equally.

If we were to take, however, $x=21$ and $y=2$, the product of which is not 36, the terms found from these, 15, 19, 23, and 27, would not answer as $27^2-15^2=504$, and $23^2-19^2=168$, which are not the values of a and b.

Exer. 38.

Here $(x-3y)+(x-y)+(x+y)+(x+3y)=a$, or

$4x=a$, and therefore $x=\frac{1}{4}a$. Also

$$(x-3y)^2+(x-y)^2+(x+y)^2+(x+3y)^3=b, \text{ or}$$
$$4x^2+20y^2=b.$$

By substituting in this $\frac{1}{4}a$ for x, and by transposition, we get $20y^2=b-\frac{1}{4}a^2$; whence $80y^2=4b-a^2$. Then, by dividing by

$80(=16\times5)$, and extracting the square root, we obtain $y=\frac{1}{4}\sqrt{\frac{4b-a^2}{5}}$. After this, the answers are found by properly combining the values of x and y.

It may be remarked, that (ALGEBRA, § 107.) the values of y might also be expressed thus $y=\frac{1}{20}\sqrt{(20b-5a^2)}$; and that a^2 cannot be greater than $4b$.

Exer. 39.

Here $(x+y)^2-(x-3y)^2=a$, or $8xy-8y^2=a$; and $(x+3y)^2-(x-y)^2=b$, or $8xy+8y^2=b$. From these two equations we get, by subtraction, $16y^2=b-a$, and therefore $y=\frac{1}{4}\sqrt{(b-a)}$. From the same equations also we get, by addition, $16xy=b+a$. Dividing these equals by $16y$, and its equal $4\sqrt{(b-a)}$, we get $x=\frac{b+a}{4\sqrt{(b-a)}}$. Lastly, putting the value of y under the form $\frac{b-a}{4\sqrt{(b-a)}}$, we readily find the values of $x-3y$, $x-y$, &c. to be the same as those given in the Answer to the Exercise.

Exer. 40.

It is plain that the length of time before twelve o'clock, at which the hands must have been last together, will be rather more than an hour; as, at 11 o'clock the hour hand points to 11 and the minute hand to 12, the minute hand having recently passed the hour hand. Let this time be x; then the space described by the minute hand in the same time will be $12x$, as it moves with twelve times the rapidity of the hour hand, describing an entire circuit while the hour hand describes only a twelfth part of a circuit. Also, in passing from one concourse to another, the minute hand must obviously have gained an entire circuit, or 12 hours. Hence $12x-x$, or $11x=12$ hours; and therefore, by dividing by 11, we get $x=1$ hour $5\frac{5}{11}$ minutes; so that the hands must have been together at $5\frac{5}{11}$ minutes before 11 o'clock.

From the foregoing solution, it is plain, that the interval between any two consecutive concourses of the hands is 1 hour $5\frac{5}{11}$ minutes. Being together, therefore, at twelve, they will be together at $5\frac{5}{11}$ minutes past one, at $10\frac{10}{11}$ minutes past two, at $16\frac{4}{11}$ minutes past three, &c.

Exer. 41.

Let x and y be the numbers, then $\frac{1}{3}x-\frac{1}{4}y=3$, and $\frac{1}{4}x+\frac{1}{5}y=10$. These equations, when cleared of fractions, may be resolved in any of the ordinary ways.

Exer. 42.

Here $\frac{1}{2}x+\frac{1}{3}y=29$, and $\frac{1}{3}x+\frac{1}{4}y=21$; and the solution is easy.

Exer. 43.

Here $x+y+z=22$, $x+1=y$, and $100x+10y+z+297=100z+10y+x$. By transposition and contraction, and by division by 99, the last of these gives $z-x=3$; and hence $z=x+3$. Then, by substituting $x+1$ for y, and $x+3$ for z in the first equation, we get $3x+4=22$; whence $x=6$: and consequently $y=x+1=7$, and $z=x+3=9$.

Exer. 44.

Let x be the shillings in one of the bank notes, and y those in the other; then, since there are 2000 shillings in 100 pounds, we have $50x+38y=2000$, and $75x+17y=2000$; the resolution of which equations will give x and y.

Exer. 45.

Let x be the number of days. Then, the spaces in miles travelled by the first, from day to day, will be 30, $30-1$, $30-2$, ...$30-(x-1)$; and the first term being 30, the last $30-(x-1)$, and the number of terms x, the sum of the series, that is, the space travelled by the first person, is (ALGEBRA, § 133.) $30x-\frac{1}{2}x(x-1)$. Now, by the question, this is equal to $20x$, the space travelled by the second person. Putting the two equal, therefore, dividing by x, and resolving the equation, we find $x=21$.

Exer. 46.

Let x be the lines in the page, and y the average number of letters in each line. Then, by the question, $(x+1)$ $(y+1)$

$-xy=96$, and $(x+2)(y+4)-xy=286$. Hence, by performing the actual multiplication, transposing, contracting, &c., we get $x+y=95$, and $2x+y=139$; whence x and y are readily found.

Exer. 47.

Let x be the money in pounds which the first person had originally, and y what the other had. Then, by the question, $x+300=3(y+300)$, or $x+300=3y+900$; and $x+800=2(y+800)$, or $x+800=2y+1600$; and x and y are easily found from these equations.

QUADRATIC EQUATIONS.

(ALGEBRA, p. 152.)

Exer. 1.

By transposing $-x^2$ and 18, we get $2x^2=162$; and from this, by dividing by 2, and extracting the square root, we get $x=\pm 9$.

Exer. 2.

By multiplying by x, and transposing $-x^2$ and -44, we obtain $2x^2=242$; and the answer is obtained by halving these equals and extracting the square root.

Exer. 3.

By squaring both members, we get $a^2+x^2=b^2x^2$. Hence, by transposing x^2, and taking the second member first, we obtain $b^2x^2-x^2=a^2$, or (§ 51.) $(b^2-1)x^2=a^2$. The values of x are then found by dividing by b^2-1, and extracting the square root.*

* If $b=\pm 1$, the value of x is infinite. It is plain, also, that if we regard $\overline{b^2-1}$ as a single quantity, we shall have (ALGEBRA, § 107.) $x=\pm\frac{a\sqrt{(b^2-1)}}{b^2-1}$.

Exer. 4.

4. By multiplying both members by $x+a$ and $x-a$, we get $(x+a)^2+(x-a)^2=b(x+a)(x-a)$; or, by actual multiplication, and by contraction, $2x^2+2a^2=bx^2-a^2b$. Hence, by transposing $2x^2$ and a^2b, and taking the second member first, we obtain $bx^2-2x^2=a^2b+2a^2$, or (ALGEBRA, § 51.) $(b-2)x^2=a^2(b+2)$; and the answer is found by dividing by $b-2$, and extracting the square root.*

Exer. 5.

Here, for removing the fractions, we must multiply by a, by $x+\sqrt{(2a^2-x^2)}$, and by $x-\sqrt{(2a^2-x^2)}$. Hence the first member becomes the product of a^2 into the sum of $x-\sqrt{(2a^2-x^2)}$ and $x+\sqrt{(2a^2-x^2)}$, which is $2x$; it is therefore simply $2a^2x$. The second member, again, is found by multiplying x into the product of $x-\sqrt{(2a^2-x^2)}$, and $x+\sqrt{(2a^2-x^2)}$; and that product (ALGEBRA, § 57.) being $2x^2-2a^2$, the second member becomes simply $2x^3-2a^2x$. Hence, therefore, by taking the second member first, we have $2x^3-2a^2x=2a^2x$; or, by transposition, and by dividing by $2x$, $x^2=2a^2$. Lastly, by extracting the square root (by § 101., ALGEBRA), we get $x=\pm a\sqrt{2}$.†

Exer. 6.

By multiplying by x^2+bx+c and by b, the given equation is changed into $bx^2+abx+b^2=ax^2+abx+ac$. Hence, rejecting abx, taking the second member first, and transposing, we get

$$ax^2-bx^2=b^2-ac, \text{ or } (a-b)x^2=b^2-ac;$$

and the answer is found by dividing by $a-b$, and extracting the square root.‡

* If $b=2$, the values of x will be infinite; but if $b=-2$, they will each be zero. Also, by multiplying the numerator and denominator of the expression for x by $\sqrt{(b-2)}$, we should have $x=\pm\frac{\sqrt{(b^2-4)}}{b-2}$.

† In addition to the values of x given in the ALGEBRA, it may also be $=0$. This arises from the circumstance, that x occurs as a multiplier in every term of the equation, $2x^3-2a^2x=2a^2x$; so that if it be $=0$, the members will become equal, each being equal to zero.

‡ In the case in which $b=a$, the value of x is infinite, unless c be also equal to a or b; and in that case x may have any value whatever, the numerator and denominator of the first member of the given equation, as well as those of its second, being equal; so that each member will be

Exer. 7, and 8.

These two exercises are wrought by means of the second rule, and they afford very simple instances of its application. The two equations differ only in the signs of their second terms, and their roots are the same, but with contrary signs.

$$x^2+6x=7$$
$$x=-3\pm\sqrt{(9+7)}$$
$$x=-3\pm4$$
$$x=1\text{, and } x=-7$$

$$x^2-6x=7$$
$$x=3\pm\sqrt{(9+7)}$$
$$x=3\pm4$$
$$x=-1\text{, and } x=7.$$

Exer. 9.

Here (2.) is derived from (1.) by multiplying by 6. Then (3.) is obtained by means of the first rule; and (4.) and (5.) exhibit the successive steps in the arithmetical computation.

$$\tfrac{1}{3}x^2-\tfrac{1}{2}x=9 \ldots\ldots\ldots\ldots (1.)$$
$$2x^2-3x=54 \ldots\ldots\ldots\ldots (2.)$$
$$x=\frac{3\pm\sqrt{(9+432)}}{4} \ldots\ldots (3.)$$
$$x=\frac{3\pm21}{4} \ldots\ldots\ldots\ldots (4.)$$
$$x=6\text{, and } x=-4\tfrac{1}{2} \ldots\ldots (5.)$$

Exer. 10.

In the work of this exercise, equation (1.) is the same as the given equation with its signs changed; and equation (2.) is derived from this by means of the second rule. The two remaining lines give the successive results of the arithmetical computation.

$$x^2-20x=-51 \ldots\ldots\ldots (1.)$$
$$x=10\pm\sqrt{(100-51)} \ldots (2.)$$
$$x=10\pm7 \ldots\ldots\ldots\ldots (3.)$$
$$x=17\text{, and } x=3. \ldots\ldots\ldots (4.)$$

equal to unity, whatever value is assigned to x. With particular numbers, one of the values of x may render the left hand member of the given equation a vanishing fraction. In such a case that member is not in its lowest terms. It may, therefore, be reduced to its lowest terms, and the solution will be obtained by means of a simple equation.

Exer. 11.

In working the exercise, equation (2.) is obtained from (1.) by multiplying by 4, and (3.) from (2.) by transposition. To find (4.), (3.) is resolved by the first rule. The rest is plain.

$$x^2-x=\tfrac{1}{4}(3x^2-7x+70)\ .\ (1.)$$
$$4x^2-4x=3x^2-7x+70\ .\ (2.)$$
$$x^2+3x=70 \ldots\ldots\ldots\ (3.)$$
$$x=\frac{-3\pm\sqrt{(9+280)}}{2}\ldots\ (4.)$$
$$x=\frac{-3\pm17}{2}\ldots\ldots\ldots\ (5.)$$
$$x=7,\text{ and } x=-10\ldots\ldots\ (6.)$$

Exer. 12.

In this operation, equation (1.) is derived from the given equation by transposition, and (2.) from (1.) by squaring. Equation (3.) is obtained from (2.) by transposition, and (4.) from (3.) by means of the second rule. The computation is easy.

$$3x-17=\sqrt{(65-x^2)}\ldots\ (1.)$$
$$9x^2-102x+289=65-x^2\ (2.)$$
$$10x^2-102x=-224\ldots\ (3.)$$
$$x=\frac{51\pm\sqrt{(2601-2240)}}{10}\ (4.)$$
$$x=\frac{51\pm19}{10}\ldots\ldots\ldots\ (5.)$$
$$x=7,\text{ and } x=3\tfrac{1}{5}\ldots\ldots\ (6.)$$

Exer. 13.

Here equation (1.) is derived from the given one by transposition, and (2.) from (1.) by squaring. From this, again, (3.) is obtained by rejecting 1, and by transposition. The values of x may then be found by means of the first rule. As, however, the absolute term is 0, and each of the other terms contains x, one value of x is 0, as that value will render the first member equal to 0, as it ought to be. Then, dividing by x, we have $x-7=0$, and thence $x=7$, the other value.

$$x-1=\sqrt{(2x^2-9x+1)}\ldots\ (1.)$$
$$x^2-2x+1=2x^2-9x+1\ .\ (2.)$$
$$x^2-7x=0\ldots\ldots\ldots\ (3.)$$
$$x=7,\text{ and } x=0\ldots\ldots\ (4.)$$

Exer. 14.

This question, which is a generalisation of the last, is wrought in the same manner. The solution of the last will be obtained from this by taking $a=2$, $b=9$, and $c=1$.*

$$x-c=\sqrt{(ax^2-bx+c^2)}$$
$$x^2-2cx+c^2=ax^2-bx+c^2$$
$$(a-1)x^2-(b-2c)x=0$$
$$x=0, \text{ and } x=\frac{b-2c}{a-1}.$$

Exer. 15.

In this solution equation (1.) is obtained from the given equation by transposing x and dividing by 4; and from this (2.) is got by means of the second rule.

$$x^2+4x=117 \ldots\ldots\ldots\ldots (1.)$$
$$x=-2\pm\sqrt{(4+117)} \ldots (2.)$$
$$x=-2\pm 11 \ldots\ldots\ldots\ldots (3.)$$
$$x=9, \text{ and } x=-13 \ldots\ldots (4.)$$

Exer. 16.

In the first of these solutions, equation (1.) is got from the given one by transposition, and (2.) from this by squaring. Then, (3.) is derived from (2.) by taking the second member first, and by transposition; and (4.) from (3.) by means of the first rule. The computation is then performed in the usual way.

The second solution is conducted according to the method pointed out in the ALGEBRA, § 154.; and it has greatly the advantage in point of brevity and facility.

$$5\sqrt{x}=22-3x \ldots\ldots\ldots\ldots (1.)$$
$$25x=484-132x+9x^2 \ldots\ldots (2.)$$
$$9x^2-157x=-484 \ldots\ldots\ldots (3.)$$
$$x=\frac{157\pm\sqrt{(157^2-17424)}}{18} \ldots (4.)$$
$$x=\frac{157\pm 85}{18} \ldots\ldots\ldots\ldots (5.)$$
$$x=13\tfrac{4}{9}, \text{ and } x=4 \ldots\ldots\ldots (6.)$$

Otherwise:

$$3x+5x^{\frac{1}{2}}=22$$
$$x^{\frac{1}{2}}=\frac{-5\pm\sqrt{(25+264)}}{6}$$
$$x^{\frac{1}{2}}=\frac{-5\pm 17}{6}$$
$$x^{\frac{1}{2}}=2, \text{ and } x^{\frac{1}{2}}=-\frac{11}{3}$$
$$x=4, \text{ and } x=\frac{121}{9}=13\frac{4}{9}.$$

* Since the second value of x is a fraction in its lowest terms, the value of that quantity will (ALGEBRA, § 129.) be indeterminate, when $a=1$ and $2c=b$. In that case, in fact, the second equation becomes identical, and x may, therefore, have any value whatever.

Exer. 17.

Here the first equation is derived from the given equation by transposition and contraction; and the solution is then obtained by means of the first rule.

$$2x^2+7x=22$$
$$x=\frac{-7\pm\sqrt{(49+176)}}{4}$$
$$x=\frac{-7\pm15}{4}$$
$$x=2, \text{ and } x=-5\tfrac{1}{2}$$

Exer. 18.

In this solution, the first equation is got from the given one by transposition; and the rest of the work proceeds according to the second rule.

$$x^2-12x=28$$
$$x=6\pm\sqrt{(36+28)}$$
$$x=6\pm8$$
$$x=14, \text{ and } x=-2$$

Exer. 19.

In this solution, equation (1.) is obtained from the given equation by multiplying by $8-x$; and (2.) from this by taking the second member first, and by transposition. The work is then carried out by means of the first rule.

$$64-x^2-26=8-x \ldots\ldots\ldots (1.)$$
$$x^2-x=30 \ldots\ldots\ldots\ldots\ldots (2.)$$
$$x=\frac{1\pm\sqrt{(1+120)}}{2}$$
$$x=6, \text{ and } x=-5$$

Exer. 20.

Here the first equation is derived from the given one by multiplying by x, and transposing; and the solution is completed by means of the first rule.

$$x^2-x=2$$
$$x=\frac{1\pm\sqrt{(1+8)}}{2}$$
$$x=2, \text{ and } x=-1$$

Exer. 21.

In this solution, equation (1.) is obtained from the given equation by multiplying by x and $x-1$; and, by transposition, and by taking the second member first, equation (1.) gives (2.). The solution is then completed by means of the first rule.

$$12x-12+10x=9x^2-9x \ldots (1.)$$
$$9x^2-31x=-12 \ldots\ldots\ldots (2.)$$
$$x=\frac{31\pm\sqrt{(961-432)}}{18}$$
$$x=\frac{31\pm23}{18}$$
$$x=3, \text{ and } x=\frac{4}{9}$$

Exer. 22.

Here the first equation is obtained from the given one by multiplying by n, and the next by means of the first rule.

$$nax^2+nx=a$$
$$x=\frac{-n\pm\sqrt{(n^2+4na^2)}}{2na}$$

Exer. 23.

In this solution the first equation is derived from the given one by multiplying by x, and by transposition; and the solution is then completed by means of the first rule.

$$x^2-ax=-1$$
$$x=\tfrac{1}{2}\{a\pm\sqrt{(a^2-4)}\}$$

Exer. 24.

Here every thing is the same as in the last exercise, except that at the conclusion the nth root must be taken.

$$x^{2n}-ax^n=-1$$
$$x^n=\tfrac{1}{2}[a\pm\sqrt{(a^2-4)}]$$
$$x=\{\tfrac{1}{2}[a\pm\sqrt{(a^2-4)}]\}^{\frac{1}{n}}$$

Exer. 25.

To solve this exercise, let the quantities be actually squared, according to what is indicated; then, by contracting and transposing, there will be obtained $x^2-8x=0$; whence $x=0$ and $x=8$.

Exer. 26.

By squaring the quantities, according to what is indicated, and by contracting and transposing, we get $x^2-6x=7$; and hence, by the second rule, we find $x=3\pm\sqrt{(9+7)}$, or $x=7$, and $x=-1$.

Exer. 27.

Here, equation (1.) is the same as the given one, but differently written. From this, equation (2.) is obtained, by means of § 154. ALGEBRA.* The rest is easy, (4.) being obtained from (3.) by cubing, and (5.) from (4.) by transposition.

$$(x+5)^{\frac{2}{3}}+3(x+5)^{\frac{1}{3}}=28 \ldots\ldots (1.)$$
$$(x+5)^{\frac{1}{3}}=\frac{-3\pm\sqrt{(9+112)}}{2} \ldots (2.)$$
$$(x+5)^{\frac{1}{3}}=4, \text{ and } (x+5)^{\frac{1}{3}}=-7 \; (3.)$$
$$x+5=64, \text{ and } x+5=-343 \ldots (4.)$$
$$x=59, \text{ and } x=-348 \ldots\ldots\ldots (5.)$$

Exer. 28.

In this exercise we have from the first equation $x=y+6$, and consequently $x^2=y^2+12y+36$. Then, by substituting this value of x^2 in the second equation, we get $2y^2+12y+36=50$; whence, by transposing, and by dividing by 2, we obtain $y^2+6y=7$; and thence, by the second rule, $y=-3\pm\sqrt{(9+7)}$, or $y=1$, and $y=-7$. Lastly, by adding 6 to each of these, we get the corresponding values, $x=7$, and $x=-1$.

A neat solution may also be obtained by squaring the members of the first equation, and taking them from those of the second, as we thus get $2xy=14$; and by adding these equals to the members of the second equation, and taking the square root, we find $x+y=\pm 8$. We then get x and y by means of § 52., ALGEBRA, p. 42.

Exer. 29.

From the first equation we have $x=5-2y$, and thence $x^2=25-20y+4y^2$. Then, by substituting for x^2 in the second equation, the value thus obtained, and by easy modifications, we get $3y^2-10y=-7$; whence, by the second rule, we find $y=\frac{5\pm\sqrt{(25-21)}}{3}$, or $y=1$ and $y=\frac{7}{3}$; and by taking the doubles of these successively from 5, we find $x=3$, and $x=\frac{1}{3}$.

* The great facility and superiority of this method would be strikingly felt by the reader, were he to attempt the solution of this exercise by first clearing the given equation of radicals.

Exer. 30.

In this solution, equation (1.) is got from the first of the given equations by transposition, and (2.) is obtained from this by squaring. We then get (3.) by substituting in the second of the given equations the value of x^2 found in (2.); and (4.) is derived from (3.) by transposition, &c. In the next place, (5.) is derived from (4.) by means of the second rule, and (6.) is the same as (5.) with some easy modifications. To get (7.), we multiply (6.) by n; and (8.) is found from (1.) and (7.); and, by easy modifications, it is transformed into (9.).*

$$x = a - ny \ldots\ldots (1.)$$

$$x^2 = a^2 - 2nay + n^2y^2 \ldots\ldots (2.)$$

$$n^2y^2 + ny^2 - 2nay + a^2 = b \ldots\ldots (3.)$$

$$(n^2+n)y^2 - 2nay = b - a^2 \ldots\ldots (4.)$$

$$y = \frac{na \pm \sqrt{\{n^2a^2 + (n^2+n)(b-a^2)\}}}{n^2+n} \ldots (5.)$$

$$y = \frac{na \pm \sqrt{\{(n^2+n)b - na^2\}}}{n^2+n} \ldots\ldots (6.)$$

$$ny = \frac{na \pm \sqrt{\{(n^2+n)b - na^2\}}}{n+1} \ldots\ldots (7.)$$

$$x = a - \frac{na \pm \sqrt{\{(n^2+n)b - na^2\}}}{n+1} \ldots\ldots (8.)$$

$$x = \frac{a \mp \sqrt{\{(n^2+n)b - na^2\}}}{n+1} \ldots\ldots (9.)$$

Exer. 31.

First Solution.

Here equation (1.) is obtained from the second of the given equations; and, by substituting the value of y thus found

* It is plain that this solution comprehends that of the last exercise, if n be taken $=2$, $a=5$, and $b=11$. It may be remarked that the roots found above apparently differ in signs from those given in the ALGEBRA. This affects nothing, however, except the *order* of the roots.

in the first of those equations, we get (2.). From this (3.) is derived by multiplying by b and x^2, and by transposition; and (4.) is then found by means of § 154. Equation (5.) is got from (1.) by squaring, and by substituting for x^2 its value according to (4.); and (6.) is derived from this by means of § 108. Finally, the values of x and y will be obtained from (4.) and (6.) by extracting the square roots.*

$$y=\frac{b}{x} \ldots\ldots\ldots\ldots\ldots (1.)$$

$$\frac{x^2}{b}+\frac{b}{x^2}=a \ldots\ldots\ldots\ldots (2.)$$

$$x^4-abx^2=-b^2 \ldots\ldots\ldots (3.)$$

$$x^2=\tfrac{1}{2}\{ab\pm\sqrt{(a^2b^2-4b^2)}\}$$

$$=\tfrac{1}{2}b\{a\pm\sqrt{(a^2-4)}\} \ldots (4.)$$

$$y^2=\frac{b^2}{\tfrac{1}{2}b\{a\pm\sqrt{(a^2-4)}\}}$$

$$=\frac{2b}{a\pm\sqrt{(a^2-4)}} \ldots\ldots (5.)$$

$$=\tfrac{1}{2}b\{a\mp\sqrt{(a^2-4)}\} \ldots (6.)$$

Second Solution.

Let $\frac{x}{y}=z$; then $\frac{y}{x}=z^{-1}$, and the first of the given equations will become $z+z^{-1}=a$; whence by multiplying by z, and by transposition, $z^2-az=-1$. The resolution of this quadratic gives z or $\frac{x}{y}=\frac{1}{2}\{a\pm\sqrt{(a^2-4)}\}$; and then the values of x and y will be found by successively multiplying and dividing the members of the second of the given equations by those of the expressions just found, and extracting the square roots.

Third Solution.

A solution preferable to either of the foregoing may be obtained in the following way. By clearing the first equation of fractions, we get $x^2+y^2=axy$, or $x^2+y^2=ab$, since $xy=b$. We have thus the sum of the squares of x and y, and their product (as in Exer. 6., ALGEBRA, p. 155.); and, therefore, by first

* The values of x and y found by this method are apparently different from those given in the ALGEBRA. They will become the same, however, if the roots be extracted according to § 169. The same principle may be employed at the close of the next solution.

adding $2xy=2b$ to $x^2+y^2=ab$, and then by subtracting it from the same, and by extracting the square roots, we get expressions for $x+y$ and $x-y$; and (§ 52.) by taking half the sum and half the difference of these, we get the values of x and y.

Exer. 32.

In this solution we get equation (1.) by transposing 3, $2(x-y)^{\frac{1}{2}}$ and y, and taking the second member first: and (2.) is obtained from this by means of § 154., and thence (3.) by squaring. We get (4.) from the expression, $x-y=9$, by transposition; and by multiplying both members of this by y, we get (5.), since by the question, $xy=36$. Equation (6.) is got by resolving (5.); and thence (7.) and (8.) are readily found. By proceeding in a similar way with the equation, $x-y=1$, we should get $y=\frac{1}{2}(-1\pm\sqrt{145})$, and $x=\frac{1}{2}(1\pm\sqrt{145})$.*

$$x-y-2(x-y)^{\frac{1}{2}}=3 \quad \ldots\ldots\ldots (1.)$$
$$(x-y)^{\frac{1}{2}}=1\pm 2=3 \text{ or } -1 \quad \ldots\ldots (2.)$$
$$x-y=9, \text{ and } x-y=1 \quad \ldots\ldots\ldots (3.)$$
$$x=y+9 \quad \ldots\ldots\ldots (4.)$$
$$xy=y^2+9y=36 \quad \ldots\ldots\ldots (5.)$$
$$y=\tfrac{1}{2}(-9\pm 15) \quad \ldots\ldots\ldots (6.)$$
$$y=3, \text{ and } y=-12 \quad \ldots\ldots\ldots (7.)$$
$$x=y+9=12, \text{ and } x=-3 \quad \ldots\ldots (8.)$$

* A neat solution would be obtained by squaring $x-y=9$, and $x-y=1$: to the results adding $4xy=144$; and extracting the square roots. By this means the sum of x and y would be known; and their difference being given, x and y themselves would be found by means of § 52. There are some differences with regard to the results found above as compared with those set down in the ALGEBRA. These arise from the different signs which the square roots may have; but they can present no difficulty, and they are, in fact, of no consequence. The solution might also be obtained by clearing the second equation of radicals, and then eliminating x or y between the result and the first equation. The equation so obtained, however, would be of the fourth degree, and could not be solved by means of the principles thus far established.

Exer. 33.

$$x^2-x-6+(x^2-x-6)^{\frac{1}{2}}=42 \ldots\ldots (1.)$$
$$(x^2-x-6)^{\frac{1}{2}}=\tfrac{1}{2}\{-1\pm\sqrt{(1+168)}\} \ldots (2.)$$
$$(x^2-x-6)^{\frac{1}{2}}=6, \text{ and } (x^2-x-6)^{\frac{1}{2}}=-7 \; (3.)$$
$$x^2-x=42, \text{ and } x^2-x=55 \ldots\ldots\ldots (4.)$$
$$x=7 \text{ or } -6, \text{ and } x=\tfrac{1}{2}(1\pm\sqrt{221}) \ldots (5.)$$

Here equation (1.) is obtained from the given equation by transposing x, and subtracting 6 from each member of the result. From (1.) we obtain (2.) and (3.) by means of § 154.; and (4.) is derived from the latter by involution and transposition. The solution is then finished by resolving the two equations contained in (4.).*

Exer. 34.

$$49(x+y)^{-2}+28(x+y)^{-1}=5 \ldots\ldots (1.)$$
$$(x+y)^{-1}=\frac{-14\pm\sqrt{(196+245)}}{49} \ldots (2.)$$
$$(x+y)^{-1}=\tfrac{1}{7}, \text{ and } (x+y)^{-1}=-\tfrac{5}{7} \ldots (3.)$$
$$x+y=7, \text{ and } x+y=-\tfrac{7}{5} \ldots\ldots\ldots (4.)$$
$$7(x-y)^{-2}-5(x-y)^{-1}=2 \ldots\ldots\ldots (5.)$$
$$(x-y)^{-1}=\frac{5\pm\sqrt{(25+56)}}{14} \ldots\ldots\ldots (6.)$$
$$(x-y)^{-1}=1, \text{ and } (x-y)^{-1}=-\tfrac{2}{7} \ldots (7.)$$
$$x-y=1, \text{ and } x-y=-\tfrac{7}{2} \ldots\ldots\ldots (8.)$$

In this operation equations (1.) and (5.) are the same as the two given equations, differently written; and (2.) and (3.), and also (6.) and (7.) are got from these by means of § 154. To get (4.) and (8.) we simply take the reciprocals of the expressions found in (3.) and (7.). In the last place, to get the values of x and y, we employ § 52., combining $x+y=7$ successively with $x-y=1$, and $x-y=-\frac{7}{2}$; and, again, combining $x+y=-\frac{7}{5}$ with each of the same values of $x-y$.

* Were the solution of this question attempted by freeing it of radicals by involution, a troublesome equation of the fourth degree would be obtained.

Miscellaneous Exercises in Quadratic Equations.

(ALGEBRA, p. 154.)

Exer. 1.

Let x be one of the numbers; then will $8-x$ be the other. By taking the sum of the squares of these and putting it equal to 50, we get $x^2-16x+64+x^2=50$; or, by transposition, contraction, and dividing by 2, $x^2-8x=-7$; the resolution of which gives $x=4\pm3$, or $x=7$, and $x=1$. By taking these severally from 8, we find the other numbers to be 1 and 7. The two numbers, therefore, are 7 and 1, and 1 and 7, so that in reality the question admits of only a single pair of answers.

For obtaining the general solution on the same principle that has been employed in the foregoing particular solution, we shall have the numbers represented by x and s_1-x; and, therefore, $x^2+(s_1-x)^2=s_2$, or, by actual involution, and by transposition, $2x^2-2s_1x=s_2-s_1^2$. Hence, by § 152.,

$$x=\tfrac{1}{2}\{s_1\pm\surd(s_1^2+2s_2-2s_1^2)\}=\tfrac{1}{2}\{s_1\pm\surd(2s_2-s_1^2)\}.$$

If x and y be assumed to denote the numbers, the solution (either particular or general) may be obtained very simply and elegantly as in the margin. In this operation, equations (1.) and (2.) express the conditions of the question; while (3.) is obtained from (1.) by squaring, and (4.) is

$$x+y=s_1 \quad \text{(1.)}$$
$$x^2+y^2=s_2 \quad \text{(2.)}$$
$$x^2+2xy+y^2=s_1^2 \quad \text{(3.)}$$
$$2xy=s_1^2-s_2 \quad \text{(4.)}$$
$$x^2-2xy+y^2=2s_2-s_1^2 \quad \text{(5.)}$$
$$x-y \text{ (or } y-x)=\surd(2s_2-s_1^2) \quad \text{(6.)}$$

derived from (2.) and (3.) by subtraction. We then get (5.) by subtraction from (2.) and (4.), and (6.) from this by extracting the square root. The required numbers will then be obtained from (1.) and (6.) by means of § 52.

Exer. 2.

Here the numbers may be represented by x and $7-x$; and by cubing these and adding the results, and by the question, we

get $343-147x+21x^2=133$; whence, by transposition, and by dividing by 21, we obtain $x^2-7x=-10$. The resolution of this equation gives $x=5$ or 2; and therefore we have $7-x$ equal to 2 or 5.

The general solution may be obtained in the same manner. Thus, let the numbers be denoted by x and s_1-x. Then, by adding the cubes of these, we get $s_1^3-3s_1^2x+3s_1x^2=s_3$; whence, by transposing s_1^3, and resolving the resulting quadratic, we get $x=\frac{3s_1^2\pm\sqrt{(12s_1s_3-3s_1^4)}}{6s_1}$; and this is easily reduced to the form given in the ALGEBRA.

Both the particular solution and the general one may be obtained very neatly and easily by means of two unknown quantities. Thus, for the general solution, we have $x+y=s_1$ and $x^3+y^3=s_3$; and by cubing the members of the first of these we get (§ 56.) $x^3+y^3+3xy(x+y)=s_1^3$, or, by substitution, $s_3+3s_1xy=s_1^3$; whence $xy=\frac{s_1^3-s_3}{3s_1}$. We have now, therefore, the sum and product of x and y; and their difference will be obtained by taking four times the product from the square of the sum, and extracting the square root; and lastly, x and y will be found by means of § 52. See Exam. 6., ALGEBRA, p. 140.

The solution may also be readily obtained by dividing the members of $x^3+y^3=s_3$ by those of $x+y=s_1$; as (§ 61.) the quotient is $x^2-xy+y^2=\frac{s_3}{s_1}$. Then, from this and from the square of $x+y=s_1$, we get, by subtraction, $3xy=s_1^2-\frac{s_3}{s_1}$; and the rest of the solution may be readily obtained in different ways.*

* If the sum of the required numbers be put $=2s$ and their difference $=2x$, the numbers will be $s+x$ and $s-x$; and, by taking the sum of the cubes of these, we get $2s^3+6sx^2=s_3$; whence x will be obtained by the resolution of a simple quadratic. In like manner, in the next question, if the difference be put $=2d$, and the sum $=2x$, the numbers will be $x+d$ and $x-d$; and by taking the difference of the cubes of these, we obtain $6dx^2+2d^3=d_3$, which is also a simple quadratic.

Exer. 3.

Here we may denote the required numbers by x and $x+2$ (or with equal propriety and facility by x and $x-2$). Then, by cubing these, and by subtraction, we get $6x^2+12x+8=218$; whence, by transposing 8, and by dividing by 6, we obtain $x^2+2x=35$; and thence $x=5$ and $x=-7$, and consequently $x+2=7$, and $x+2=-5$.

For the general solution, we may denote the numbers by x and $x+d_1$; and, by taking the difference of the cubes of these, we obtain $3d_1x^2+3d_1^2x+d_1^3=d_3$, — a quadratic, the resolution of which presents no difficulty.

The solution of this question, like that of the last, may be easily effected by means of two unknown quantities. Thus, we have $x-y=d_1$, and $x^3-y^3=d_3$; and, as in the last, the solution may be effected by the use of either § 56. or § 61., ALGEBRA, p. 44. and 46.

Exer. 4.

This will be solved in the same manner as Exam. 6., ALGEBRA, page 140. See also the conclusion of the solution given above for Exer. 2.

Exer. 5.

Here, by assuming x and $x+5$ (it would be equally proper to assume x and $x-5$), we have, by the question, $x(x+5)$ or $x^2+5x=36$. The resolution of this gives $x=4$ and $x=-9$; and consequently $x+5=9$, and $x+5=-4$.

For a general solution let x and $x+d$ represent the numbers. Then $x(x+d)$ or $x^2+dx=p$; the resolution of which gives $x=\frac{1}{2}\{-d\pm\sqrt{(d^2+4p)}\}$: and hence $x+d=\frac{1}{2}\{d\pm\sqrt{(d^2+4p)}\}$.

Both the particular solution and the general one will be readily obtained by denoting the required numbers by characters representing two unknown quantities. We thus have $x-y=d$, and $xy=p$; and if four times the latter be added to the square of the former, and the square roots of the equals thus obtained be taken, we shall have $x+y=\sqrt{(d^2+4p)}$; and the values of x and y will be obtained by means of § 52.

Exer. 6.

Let x be one of the numbers; then $\surd(85-x^2)$ will be the other; and by the question, $x\surd(85-x^2)=18$. Hence, by squaring both members and changing the signs, we get $x^4-85x^2=-324$; and from this, by § 154., we find

$$x=\sqrt{\tfrac{1}{2}\{85\pm\surd(7225-1296)\}}=\pm 9 \text{ or } \pm 2;$$

and therefore $\surd(85-x^2)=\pm 2$ or ± 9. The general solution on this principle would proceed similarly.

Otherwise.

Let x and y be the numbers: then $xy=p$, and $x^2+y^2=s_2$. Let the latter be successively increased and diminished by twice the former; then, by extraction, there will be obtained $x+y=\surd(s_2+2p)$, and $x-y$ (or $y-x$)$=\surd(s_2-2p)$; and x and y will be found by means of § 52.

Exer. 7.

Let x and $40x^{-1}$ represent the numbers; then, by the question, $x^2-1600x^{-2}=39$; whence, by multiplying by x^2 and transposing, we obtain $x^4-39x^2=1600$. From this we get, by means of § 154., $x=\pm 8$, and $x=\pm 5\sqrt{-1}$; and, by dividing 40 by these, we find the corresponding values of the other number to be ± 5, and $\mp 8\sqrt{-1}$.

For getting the general solution in a similar manner, we assume x and px^{-1} as the numbers, and we have $x^2-p^2x^{-2}=d_2$. Then, by multiplying by x^2 and transposing, we find $x^4-d_2x^2=p^2$; and the solution is easily effected by means of §§ 154. and 108.

The solution might also be obtained by assuming x and xy to denote the numbers; and for another method still, we might represent the numbers by x and y. In the latter method, we should have $xy=p$, and $x^2-y^2=d_2$; whence, by squaring we get $x^2y^2=p^2$, and $x^4-2x^2y^2+y^4=d_2{}^2$. Then, by adding to the latter $4x^2y^2=4p^2$ (four times the former), and extracting the square root, we get $x^2+y^2=\surd(d_2{}^2+4p^2)$. The rest is easy.*

* This question affords an instance in which no advantage is gained by assuming $x+y$ and $x-y$ to represent the required numbers. Since,

Exer. 8.

If x and y be taken to denote the required numbers, the solution will be easily and simply obtained as in the margin; where equation (1.) is got by cubing $x+y$, and subtracting x^3+y^3 from the result;

$$\left.\begin{aligned} &3x^2y+3xy^2=a \ldots\ldots\ldots\ldots\ldots\ldots (1.)\\ &3x^2y-3xy^2=b \ldots\ldots\ldots\ldots\ldots\ldots (2.)\\ &6x^2y=a+b, \text{ and } 6xy^2=a-b \ldots\ldots\ldots (3.)\end{aligned}\right\}$$

$$\frac{x}{y}=\frac{a+b}{a-b} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

$$6x^3=\frac{(a+b)^2}{a-b}, \text{ and } 6y^3=\frac{(a-b)^2}{a+b} \ldots \quad (5.)$$

and (2.) by taking the cube of $x-y$ from x^3-y^3. The two expressions in line (3.) are got from (1.) and (2.) by addition and subtraction; and (4.) is found by dividing the members of the first equation in (3.) by those of the second. The first of the equations in line (5.) is obtained from (4.) and the first equation in (3.), by multiplication; and the next from (4.) and the second in (3.), by division. In the last place, the answers will be found from the equations in (5.) by dividing by 6, and extracting the cube roots.

The particular solution would be had by using 60 and 36 instead of a and b.

Exer. 9.

To solve this question in perhaps the easiest manner, let x and y be taken to denote the required numbers: then $xy=6$ and $x^3+y^3=35$. From the square of the latter take four times the cube of the former, and by extracting the square roots of the remainder, there will be obtained $x^3-y^3=19$. Lastly, take half the sum and half the difference of the members of this equation, and those of $x^3+y^3=35$, and extract the cube roots.*

however, the *difference of their squares* is given, we might assume with advantage $2x$ to denote the *sum of their squares*. We should then have the two numbers represented by $\surd(x+\frac{1}{2}d_2)$ and $\surd(x-\frac{1}{2}d_2)$; and we should obtain a very easy solution by putting the product of the squares of these equal to p^2.

* From the equation $xy=6$, we might find a value for x or y; and by substituting this in the other equation, $x^3+y^3=35$, we should get an equation which would be resolved by means of § 154. A similar remark is applicable with regard to the next exercise.

Exer. 10.

By assuming x and y to represent the required numbers, we shall have $xy=20$, and $x^3-y^3=61$; and the solution will proceed exactly as that of the preceding exercise, except that we are to *add* $4x^3y^3=4\times 20^3$ to the squares of the members of the second equation.

Exer. 11.

By putting x and y to denote the numbers, we have, by the question,

$xy=240$, and $(x+4)(y-3)$ or $xy+4y-3x-12=240$.

Hence, by subtraction, we get $4y-3x-12=0$. Multiply this by x, and take the product from $4xy=240\times 4=960$, and there will remain $3x^2+12x=960$. By dividing this by 3, we get $x^2+4x=320$; and by resolving this last result, we find $x=16$, and $x=-20$. Lastly, by dividing 240 by these successively, we find the corresponding values of y to be 15 and -12.

By changing the signs of x and y in the two original equations, and by a few obvious modifications, it is easy to show that the negative values, -20 and -12, taken positively, are the answers to the analogous question in which one of the numbers is *diminished* by 4, and the other *increased* by 3.

For obtaining a general solution, we have $xy=a$, and $(x+b)(y-c)$ or $xy+by-cx-bc=a$; and by subtraction, $by-cx-bc=0$. We may then multiply this by x; and by taking the result from $bxy=ab$, we get $cx^2+bcx=ab$; the resolution of which equation gives $x=\dfrac{-bc\pm\sqrt{(b^2c^2+4abc)}}{2c}$. The value of y is readily found by dividing a by this value of x, and for a simplification, by multiplying the terms of the resulting fraction by $bc\pm\sqrt{(b^2c^2+4abc)}$, according to § 108.

Exer. 12.

Here, by representing the required numbers by $x-3y$, $x-y$, $x+y$, and $x+3y$, we have the product of the means $=x^2-y^2$,

and that of the extremes $=x^2-9y^2$. Taking the latter from the former, we get, by the question, $8y^2=8$, and consequently $y=\pm 1$. Hence, the foregoing products become x^2-1 and x^2-9; and, by the question, the product of these is 105; that is, $x^4-10x^2+9=105$. Then, by transposing 9, and by means of § 154., we get $x=\pm 4$, and $x=\pm\sqrt{-6}$, the latter of which is to be rejected. Hence, taking x positive, and y first positive, and then negative, we find the required numbers to be 1, 3. 5, and 7, or 7, 5, 3, and 1; and again, by taking x negative, and y first positive and then negative, we find the numbers to be -7, -5, -3, and -1, and -1, -3, -5, and -7.

For the general solution, the products of the means and extremes are, as before, x^2-y^2 and x^2-9y^2. Then, by taking the product and the difference of these, we get $x^4-10y^2x^2+9y^4=a$, and $8y^2=b$. The latter of these gives $y^2=\frac{1}{8}b$; and from the former, and by means of § 154., we get $x^2=5y^2\pm\sqrt{(16y^4+a)}$. This latter becomes $x^2=\frac{5}{8}b\pm\sqrt{(\frac{1}{4}b^2+a)}$, by the substitution of $\frac{1}{8}b$ for y^2: whence, by easy modifications, we get

$$x^2=\tfrac{5}{8}b\pm\tfrac{1}{2}\sqrt{(b^2+4a)}=\tfrac{10}{16}b\pm\tfrac{8}{16}\sqrt{(b^2+4a)};\text{ and therefore,}$$

$$x=\pm\tfrac{1}{4}\sqrt{\{10b\pm 8\sqrt{(b^2+4a)}\}}.$$

Exer. 13.

Putting x to denote the number of oxen purchased, we have, by the question, $\frac{120}{x}-\frac{120}{x+3}=2$. Dividing by 2, clearing the result of fractions, and simplifying the equation so obtained, we get $x^2+3x=180$; an equation which gives $x=\frac{1}{2}(-3\pm 27)$, or $x=12$, and $x=-15$.

To interpret the latter result, change x into $-x$ in the original equation: then, after some slight modifications, we get $\frac{120}{x-3}-\frac{120}{x}=2$; which shows that the root -15, taken positively, is the answer to the question in which, had the purchaser got 3 oxen fewer than he did, the price of each would have been increased by £2, which is the next question.

Exer. 14.

Here we have $\frac{120}{x-3} - \frac{120}{x} = 2$; and, by a process similar to that which was employed in the solving of the last exercise, we find $x=15$, and $x=-12$.

It is easy to show, as in the last solution, that the negative root -12, taken positively, is the answer to Exer. 13.

Exer. 15.

Putting x to denote the number of persons present, we have $\frac{144}{x-2} - \frac{144}{x} = 1$, (144 being the shillings in seven pounds four shillings). Freeing this of fractions, &c., we get $x^2-2x=288$; whence $x=18$, and $x=-16$.

To interpret the negative root, let x be taken negative in the original equation, and it will become $\frac{144}{x} - \frac{144}{x+2} = 1$; which shows, that if two more than x persons had paid their quotas, the payment of each of the others would thus have been diminished by a shilling; and hence it follows, that 16 is the answer to the next question.

Exer. 16.

Here we have $\frac{144}{x} - \frac{144}{x+2} = 1$; the resolution of which gives $x=16$, and $x=-18$: and hence it may be shown by an easy interpretation, that 18 must be the answer to the preceding exercise.

Exer. 17.

Let x be the number of days travelled by the first person, and $x-2$ will be the number travelled by the second. Then, the distances travelled by the first person on the several successive days will be 10 miles, 11, 12, &c.; the first term being 10, the

common difference 1, and the number of terms x. According, therefore, to § 133., the whole space travelled by him will be $10x+\frac{1}{2}x(x-1)$ or $\frac{1}{2}(x^2+19x)$; and the space travelled by the other is $15(x-2)$ or $15x-30$. Now, by the question, the sum of these spaces is 300 miles; that is $\frac{1}{2}(x^2+19x)+15x-30=300$. Hence, by doubling, transposing, and contracting, we get $x^2+49x=660$; and thence $x=\frac{1}{2}(-49\pm71)$, or $x=11$, and $x=-60$. Then, by taking $x=11$, $\frac{1}{2}(x^2+19x)$ becomes 165, the space required.*

Exer. 18.

To give a general solution of this question, let $35=a$ and $30=b$; and let the numbers be represented by x, xy, xy^2, and xy^3. Then, by the question, $xy^3+x=a$ and $xy^2+xy=b$. Divide the former of these by the latter; then

$$\frac{y^3+1}{y^2+y}=\frac{a}{b}, \text{ or } \frac{y^2-y+1}{y}=\frac{a}{b},$$

by dividing the numerator and denominator of the first member by $y+1$. By clearing this of fractions and transposing, we get $by^2-ay-by=-b$, whence

$$y=\frac{a+b\pm\sqrt{\{(a+b)^2-4b^2\}}}{2b}=\frac{a+b\pm\sqrt{\{(a-b)(a+3b)\}}}{2b}.$$

Hence, by restoring the values a and b, we get $y=\frac{3}{2}$, and $y=\frac{2}{3}$. These values of y will give the same numbers for the answer,

* It would appear from a little consideration in reference to the root -60, that, under the same law of movement, the two travellers would have been together 60 days before the one started from A, and 62 before the other set out from B, and that the point where they would have been together would have been in the line AB continued through B, and at the distance of 930 miles from B. Then, on starting from that point, one of them would move towards B at the uniform rate of 15 miles each day, and would reach B in 62 days; while the other, for the first 50 days, would move in the same direction, the first day 50 miles, the next 49, the next 48, &c.; and on the fifty-first day he would rest; then, during the next nine days he would move in the opposite direction, one mile the first, two the second, and so on; and at the end of the whole sixty days, he would be at A.

only in a reversed order. The value of x is readily found from the equation, $xy^2+xy=b=30$, to be 8 (or 27); and the required numbers are 8, 12, 18, and 27.

Exer. 19.

What is given in the note in page 156. of the ALGEBRA, is sufficient as a solution of this exercise.

The question will be easily solved also by equalling the products of the extremes and means of the analogies in the same note, and dividing the members of one of the equations thus obtained by those of the other. By this means we get

$$\frac{x-42}{x+42}=\frac{9}{16}\cdot\frac{x+42}{x-42}; \text{ whence } 16(x-42)^2=9(x+42)^2;$$

and, therefore, $4(x-42)=\pm3(x+42)$, an equation from which the values of x are easily found to be 294 and 6.

(ALGEBRA, p. 165.)

Exer. 1.

$$\sqrt{(8\pm\sqrt{60})}=\sqrt{\tfrac{1}{2}\{8+\sqrt{(64-60)}\}}\pm\sqrt{\tfrac{1}{2}\{8-\sqrt{(64-60)}\}}$$
$$=\sqrt{5}\pm\sqrt{3}.$$

Exer. 2.

$$\sqrt{(6\pm2\sqrt{5})}=\sqrt{\tfrac{1}{2}\{6+\sqrt{(36-20)}\}}\pm\sqrt{\tfrac{1}{2}\{6-\sqrt{(36-20)}\}}$$
$$=\sqrt{5}\pm1.$$

Exer. 3.

$$\sqrt{(49\pm12\sqrt{13})}=\sqrt{\tfrac{1}{2}\{49+\sqrt{(2401-1872)}\}}\pm$$
$$\sqrt{\tfrac{1}{2}\{49-\sqrt{(3401-1872)}\}}=\sqrt{\tfrac{1}{2}(49+23)}\pm\sqrt{\tfrac{1}{2}(49-23)}$$
$$=6\pm\sqrt{13}.$$

Exer. 4.

$$\sqrt{(76 \pm 32\sqrt{3})} = \sqrt{\tfrac{1}{2}\{76 + \sqrt{(5776 - 3072)}\}} \pm \sqrt{\tfrac{1}{2}\{76 - \sqrt{(5776 - 3072)}\}} = \sqrt{\tfrac{1}{2}(76 + 52)} \pm \sqrt{\tfrac{1}{2}(76 - 52)}$$
$$= 8 \pm \sqrt{12} = 8 \pm 2\sqrt{3}.$$

Exer. 5.

$$\sqrt{(39 \pm 6\sqrt{42})} = \sqrt{\tfrac{1}{2}\{39 + \sqrt{(1521 - 1512)}\}} \pm \sqrt{\tfrac{1}{2}\{39 - \sqrt{(1521 - 1512)}\}} = \sqrt{\tfrac{1}{2}(39 + 3)} \pm \sqrt{\tfrac{1}{2}(39 - 3)}$$
$$= \sqrt{21} \pm 3\sqrt{2}.$$

Exer. 6.

$$\sqrt{(52 \pm 30\sqrt{3})} = \sqrt{\tfrac{1}{2}\{52 + \sqrt{(2704 - 2700)}\}} \pm \sqrt{\tfrac{1}{2}\{52 - \sqrt{(2704 - 2700)}\}} = \sqrt{\tfrac{1}{2}(52 + 2)} \pm \sqrt{\tfrac{1}{2}(52 - 2)}$$
$$= \sqrt{27} \pm \sqrt{25} = 3\sqrt{3} \pm 5.$$

Exer. 7.

$$\sqrt{\{p^2 + 2\sqrt{(p^2q^2 - q^4)}\}} = \sqrt{\tfrac{1}{2}\{p^2 + \sqrt{(p^4 - 4p^2q^2 + 4q^4)}\}} + \sqrt{\tfrac{1}{2}\{p^2 - \sqrt{(p^4 - 4p^2q^2 + 4q^4)}\}} = \sqrt{(p^2 - q^2)} + q.$$

Exer. 8.

By taking from the square of $4a^2$ the square of what follows it, and extracting the square root, we obtain $4a^2 - 8ab + 2b^2$. Then, by taking half the sum and half the difference of this and $4a^2$, extracting the square roots, and connecting them by the sign $-$, we get the answer.

Exer. 9.

From the square of $2a^2 + 2b^2$, take the square of what follows it; then, by extracting the square root of the remainder, there is obtained $2ab$; and the answer is found by taking half the sum and half the difference of this and $2a^2 + 2b^2$, and connecting the square roots of the results by the sign $\pm$.

General Theory and Resolution of Equations.

(ALGEBRA, p. 170.)

Exer. 1.

Multiply $x-3$ by $x+2$, and put the product$=0$.

Exer. 2.

Here, as the factors are $x-2-3\sqrt{-1}$ and $x-2+3\sqrt{-1}$, they are the difference and the sum of $x-2$ and $3\sqrt{-1}$; and therefore (ALGEBRA, § 57.) their product is the difference of the squares of these. Accordingly, from x^2-4x+4, the square of the first, we take -9, the square of the second, and we put the result $=0$.

Exer. 3.

Here, $(x-1)(x-2)(x-3)=0$. In this and in all similar instances, the actual multiplication is most quickly and easily performed by the method of detached coefficients.

Exer. 4.

Put the actual product of $x-2$, $x+4$, $x-5$, and $x+6=0$.

Exer. 5.

Here, (by ALGEBRA, § 57.) $(x-1)(x+1)=x^2-1$, and $(x-\sqrt{-1})(x+\sqrt{-1})=x^2-(-1)=x^2+1$. Lastly,

$$(x^2-1)(x^2+1)=0, \text{ or } x^4-1=0.$$

Exer. 6.

Here, the answer is obtained by the actual expansion and multiplication of $(x-1)^3(x+1)(x+2)=0$.

Transformation of Equations.

(ALGEBRA, p. 174.)

Exer. 7.

By substituting $x'-a$ for x, we get $(x'-a)^2+2a(x'-a)-b=0$: whence, by actually performing the operations that are indicated, and by contracting, we obtain $x'^2-a^2-b=0$.

Exer. 8.

Here, we assume $x=x'-2$; and by squaring and cubing, we get $x^2=x'^2-4x'+4$, and $x^3=x'^3-6x'^2+12x'-8$. Then the answer is obtained by substituting these in the given equation.

Exer. 9.

We are here to assume $x=x'+\frac{2}{3}$; of which the square is $x^2=x'^2+\frac{4}{3}x'+\frac{4}{9}$, and the cube $x^3=x'^3+2x'^2+\frac{4}{3}x'+\frac{8}{27}$. The answer is then found by substituting these values of x^3 and x^2 in the given equation.

Exer. 10.

By assuming $x=x'+2$, we find $x^3=x'^3+6x'^2+12x'+8$, and $x^4=x'^4+8x'^3+24x'^2+32x'+16$; and the answer is obtained by substituting these in the given equation.

(ALGEBRA, p. 177.)

Exer. 11.

Here, by following the method pointed out in the ALGEBRA, p. 175. and 176. we find the coefficients of the transformed equation to be 1, 0, and -1; and it is, therefore, $x'^2-1=0$.

$$\begin{array}{rrr|l} 1 & -4 & 3 & 2 \\ & 2 & -4 & \\ \hline & -2 & -1 & \\ & 2 & & \\ \hline & 0 & & \end{array}$$

Exer. 12.

Here, by a like process, the coefficients of the required equation are found to be 1, 12, 35, and 0. The equation, therefore, is

$$x'^3+12x'^2+35x'=0$$

$$\begin{array}{rrrr|r}
1 & 0 & -13 & -12 & 4 \\
 & 4 & 16 & 12 & \\
\hline
 & 4 & 3 & 0 & \\
 & 4 & 32 & & \\
\hline
 & 8 & 35 & & \\
 & 4 & & & \\
\hline
 & 12 & & &
\end{array}$$

Exer. 13.

We here get, by the usual process, 1, 20, 150, 375, and 4, as the coefficients of the required equation. It is, accordingly,

$$x'^4+20x'^3+150x'^2+375x'+4=0.$$

$$\begin{array}{rrrrr|r}
1 & 0 & 0 & -125 & 4 & 5 \\
 & 5 & 25 & 125 & 0 & \\
\hline
 & 5 & 25 & 0 & 4 & \\
 & 5 & 50 & 375 & & \\
\hline
 & 10 & 75 & 375 & & \\
 & 5 & 75 & & & \\
\hline
 & 15 & 150 & & & \\
 & 5 & & & & \\
\hline
 & 20 & & & &
\end{array}$$

Exer. 14.

Here we use -1, because the roots are to be increased. The coefficients are found to be 1, 0, -3, and -1; and therefore the required equation is

$$x'^3-3x'-1=0.$$

$$\begin{array}{rrrr|r}
1 & 3 & 0 & -3 & -1 \\
 & -1 & -2 & 2 & \\
\hline
 & 2 & -2 & -1 & \\
 & -1 & -1 & & \\
\hline
 & 1 & -3 & & \\
 & -1 & & & \\
\hline
 & 0 & & &
\end{array}$$

Resolution of Equations by Horner's method.

(ALGEBRA, p. 192.)

Exer. 1.

Resolve the equation, $x^3+7x-3=0$.

00	700	-3000	0·418128
4	16	2864	
4	716	$-{}_{2}136000$	
4	32	74921	
8	$_{2}74800$	$-{}_{3}610790$	
4	121	601128	
$_{2}120$	74921	$-{}_{4}9662$	
1	122	7524	
121	$_{3}75043$	-2138	
1	98	1505	
122	75141	-633	
1	98	602	
$_{3}123$	$_{4}75239$	-31	
. .	. . .		

In this operation, the work is shortened by adding only one cipher, instead of three, in the line marked with $_3$ subscribed in the third column; and, accordingly, none is annexed in the corresponding line in the second column, and one figure is cut off from the like line in the first.

If the first member of the given equation be divided by $x-0·418128$, the quotient is $x^2-0·418128x+7·174831$; and by putting this equal to zero, we get a quadratic, the roots of which are imaginary; and these are the remaining roots of the proposed equation.

We may arrive at the same conclusions by means of Sturm's Theorem (ALGEBRA, p. 287.). In employing this theorem, we shall have, in the present instance, —

$$X=x^3+7x-3; \qquad X_2=-14x+9; \text{ and}$$
$$X_1=3x^2+7; \qquad X_3=-1615.$$

Hence the signs will be,

for $x=-\infty$, $-$, $+$, $+$, and $-$; and
for $x=\infty$, $+$, $+$, $-$, and $-$.

We have, therefore, $m_1=2$, and $m_2=1$; the difference of which is 1: wherefore there is only one real root; and by taking x successively $=0$ and $x=\infty$, we find that it must be positive.

Exer. 2.

Resolve the equation, $x^3-2x^2+3x-4=0$.

Here (§ 187.) the greatest root cannot exceed 5; and, by Descartes's rule, there can be no negative root. By trying numbers, therefore, between 0 and 5, we find that a root lies between 1 and 2; and the work for finding it is as follows:

-2	3	-4	$1{\cdot}6506292$, nearly.
1	-1	2	
-1	2	$-_{2}2000$	
1	0	1776	
0	$_{2}200$	$-_{3}224000$	
1	96	221125	
$_{2}10$	296	$-_{4}287500$	
6	132	274158	
16	$_{3}42800$	$-_{5}13342$	
6	1425	9142	
22	44225	-4200	
6	1450	4114	
$_{3}280$	$_{4}456750$	-86	
5	$18\cdot$	91	
285	45693		
5	18		
290	$_{5}45711$		
5	$\cdots$		
$_{4}0295$			
$\cdot\ \cdot$			

By dividing the members of the given equation by $x-1{\cdot}65063$, we get $x^2-0{\cdot}34937x+2{\cdot}42332=0$; the roots of which are plainly imaginary.

Exer. 3.

By § 178., the only real root of this equation is $-1{\cdot}6506292$, the negative of the real root of the last.

Exer. 4.

Resolve the equation, $x^4-2x^3+3x-20=0$.

Here the coefficients are 1, -2, 0, 3, and -20; and whether $+$ or $-$ be prefixed to 0, there are three changes of signs, and one permanence. Hence, by Descartes's rule, there cannot be more than three positive roots, nor more than one negative one. Also, by §§ 187. and 188., the roots lie between 21 and -6; and by some trials it will be found that one of them lies between 2 and 3 and one between -1 and -2. The computation of these is as follows: —

-2	0	3	-20	2·64868849
2	0	0	6	
0	0	3	$-_2 140000$	
2	4	8	123456	
2	4	$_2 11000$	$-_3 165440000$	
2	8	9576	135023616	
4	$_2 1200$	20576	$-_4 30416384$	
2	396	12168	27991760	
$_2 60$	1596	$_3 32744000$	$-_5 2424624$	
6	432	1011904	2112852	
66	2028	33755904	$-_6 311772$	
6	468	1025472	281856	
72	$_3 249600$	$_4 34781376$	$-_7 29916$	
6	3376	20832 ·	28187	
78	252976	3498970	-1729	
6	3392	20880	1409	
$_3 840$	256368	$_5 3519850$	-320	
4	3408	157 ·	317	
844	$_4 259776$	352142	3	
4	6 ·	157		
848	2604	$_6 352299$		
4	6	2 ·		
852	2610	35232		
4	6	2		
$_4 856$	$_6 2616$	$_7 35234$		
·	· ·	· · ·		

-2	0	3	-20	$-1{\cdot}87683707$
-1	3	-3	0	
-3	3	0	$-{}_2200000$	
-1	4	-7	167616	
-4	7	$-{}_27000$	$-{}_3323840000$	
-1	5	-13952	293511561	
-5	${}_21200$	-20952	$-{}_430328439$	
-1	544	-18816	26599062	
$-{}_260$	1744	$-{}_339768000$	-3729377	
-8	608	-2162223	3564120	
-68	2352	-41930223	-165257	
-8	672	-2207989	133734	
-76	${}_3302400$	$-{}_444138212$	-31523	
-8	6489	$-19356\cdot$	31215	
-84	308889	-4433177	-308	
-8	6538	-19392	312	
$-{}_3920$	315427	-4452569		
-7	6587	$-258\cdot$		
-927	${}_4322014$	-445515		
-7	$6\cdot$	-258		
-934	3226	-445773		
-7	6	$-1\cdot$		
-941	3232	-44578		
7	6	-1		
$-{}_4948$	3238	-44579		
$\cdot$	$\cdot\ \cdot$	$\cdot\ \cdot\ \cdot$		

The factors corresponding to the roots that have now been found are $x-2{\cdot}648688$ and $x+1{\cdot}876837$; the product of which is $x^2-0{\cdot}771851\,x-4{\cdot}971156$: and, by dividing the given equation by this, we get $x^2-1{\cdot}228149x+4{\cdot}023209=0$ *; a

* It is easy to see that the coefficient of x in this quotient will be found by taking $-0{\cdot}771851$ from the coefficient -2; and that $4{\cdot}023209$ will be got by dividing -20 by $-4{\cdot}971156$: and a complete verification of the correctness of the work would be obtained by finding the product of the two quadratic factors just found, as it ought to be the same as the first member of the given equation, except some trifling differences at the end of the decimals. It is plain that the easy method here pointed out for finding the quadratic which involves the remaining roots, may be employed in all similar cases.

quadratic equation, the resolution of which will give the two remaining roots. Both of these, however, are evidently imaginary, since the square of the coefficient of x is less than four times 4·023209.

Exer. 5.

Resolve the equation, $x^4-2x^3-3x^2-4x+5=0$.

According to Descartes's rule, this equation cannot have more than two positive roots, nor more than two negative ones; and by §§ 187. and 188., Algebra, pp. 181, 182., the roots must lie between 6 and -3. Taking, therefore, x successively equal to 6, 0, and -3, we get 737, 5, and 125 as the corresponding values of fx; and hence, according to the Algebra, § 184., there must be either two real positive roots, or none, between 0 and 5, and either two real negative ones, or none, between 0 and -3. If we now take $x=2$, we get $fx=-15$; and therefore (Algebra, § 184.), one of the positive roots must lie between 0 and 2, and the other between 2 and 4: and some farther trials will show that one of them lies between 0 and 1, and the other between 3 and 4, so that their first figures are 0 and 3. The operations for finding them are as follows:—

-20	-300	-4000	$50000 \mid 0{\cdot}7287268$
7	-91	-2737	-47159
-13	-391	-6737	$_228410000$
7	-42	-3031	-19705744
-6	-433	$-_29768000$	$_38704256$
7	7	-84872	-7976864
1	$-_242600$	-9852872	$_4727392$
7	164	-84536	-700539
$_280$	-42436	$-_39937408$	26853
2	168	$-3367\cdot$	-20021
82	-42268	-997108	6832
2	172	-3367	-6006
84	-0_342096	$-_41000475$	826
2	$\cdot\ \cdot\ \cdot$	$-29\cdot$	-801
86		-100077	25
2		-29	
0_388		-100106	
$\cdot$		$\cdot\cdot\cdot$	

-2	-3	-4	$5 \mid 3{\cdot}18247782$
3	3	0	-12
1	0	-4	$-{}_{2}70000$
3	12	36	35401
4	12	${}_{2}32000$	$-{}_{3}345990000$
3	21	3401	334846976
7	${}_{2}3300$	35401	$-{}_{4}11143024$
3	101	3503	8990510
${}_{2}100$	3401	${}_{3}38904000$	$-{}_{5}2152514$
1	102	2951872	1801808
101	3503	41855872	$-{}_{6}350706$
1	103	3019456	315441
102	${}_{3}360600$	${}_{4}44875328$	-35265
1	8384	7722·	31545
103	368984	4495255	-3720
1	8448	7726	3605
${}_{3}1040$	377432	${}_{5}4502981$	-115
8	8512	154·	90
1048	${}_{4}385944$	450452	-25
8	2·	154	
1056	3861	${}_{6}450606$	
8	2	2·	
1064	3863	45063	
8	2	2	
${}_{4}1072$	3865	45065	
·	· ·	· · ·	

Here, $x-0{\cdot}728727$ and $x-3{\cdot}182478$ are the factors corresponding to the roots which we have found. The product of these is $x^2-3{\cdot}911205x+2{\cdot}319158$; and by dividing the given equation by this, we get $x^2+1{\cdot}911205x+2{\cdot}155955=0$. The roots of this, which are easily found, are both imaginary.

Exer. 6.

Resolve the equation, $x^5-7x^4+15x^3-58x^2+44x-300=0$.

In this equation there are no permanences, and therefore there are no negative roots; but, according to § 187., there may be one, three, or five roots between 0 and 301; limits which are so wide

as to be of scarcely any use. It is easy to see, also, that x must be far less than the superior one of these limits; and it will be found by a few trials that there is a root between 6 and 7. The computation of this root will be as follows: —

−7	15	−58	44	−300	6·1195379
6	−6	54	−24	120	
−1	9	−4	20	$-{}_{2}18000000$	
6	30	234	1380	14880931	
5	39	230	${}_{2}14000000$	$-{}_{3}3119069$	
6	66	630	880931	1587560	
11	105	${}_{2}860000$	14880931	$-{}_{4}1531509$	
6	102	20931	902094	1444689	
17	${}_{2}20700$	880931	${}_{3}15783025$	$-{}_{5}86820$	
6	231	21163	9257·	80705	
${}_{2}230$	20931	902094	1587560	$-{}_{6}6115$	
1	232	21396	9279	4844	
231	21163	${}_{3}923490$	${}_{4}1596839$	−1271	
1	233	22·	837·	1130	
232	21396	9257	160521	−141	
1	234	22	837	145	
233	$0_{3}21630$	9279	${}_{5}161358$		
1	· ·	22	5·		
234		${}_{4}9301$	16141		
1		· ·	5		
$0_{3}235$			16146		
·			· · ·		

Hence, $x-6{\cdot}119538$ is a factor of the given equation; and, by dividing by it, we get

$$x^4-0{\cdot}880462x^3+9{\cdot}611979x^2+0{\cdot}820871x+49{\cdot}023351=0.$$

Now, this equation can have no negative roots, as we saw that the original equation could have none: and (by Descartes's rule), as there are only two variations of signs, there cannot be more than two real positive roots; so that at least two of the roots must be imaginary. Should there be real roots, they must lie (ALGEBRA, § 187.) between 0 and 0·880462. By trying num-

bers between these limits, however, such as 0·1, 0·2, 0·5, &c.*, we shall find that each of them gives a result not differing much from 49, none of the results making any approach to zero; and we conclude accordingly, that the equation has only one real root, — the one already found.

Exer. 7.

Resolve the equation, $x^2+x-1=0$.

This equation, it is easily found, has a positive root beginning with zero, and a negative one beginning with -1. The work by Horner's method is as follows: —

10	−100 | 0·618034	1	−1 | −1·618034
6	96	−1	0
16	$-{}_{2}400$	0	$-{}_{2}100$
6	221	−1	96
${}_{2}220$	$-{}_{3}17900$	$-{}_{2}10$	$-{}_{3}400$
1	17824	−6	221
221	$-{}_{4}7600$	−16	$-{}_{4}17900$
1	6708	−6	17824
${}_{3}2220$	−892	$-{}_{3}220$	$-{}_{5}760$
8	894	−1	671
2228		−221	−89
8		−1	89
${}_{4}22360$		$-{}_{4}2220$	
..		−8	
		−2228	
		−8	
		$-{}_{3}2236$	
		..	

* For avoiding trouble in trying tenths by means of detached coefficients, the tenths should be taken as units, and the coefficients of fx written −70, 1500, −58000, 440000, and −30000000, and then five places of decimals should be pointed off in the number found in the last

Exer. 8.

Resolve the equation $x^3+x^2+x-1=0$.

Here there is one positive root between 0 and 2, and its first figure is readily found to be 0.

10	100	-1000	0·543689·
5	75	875	
15	175	$-_{2}125000$	
5	100	114064	
20	$_{2}27500$	$-_{3}1093600$	
5	1016	888801	
$_{2}250$	28516	$-_{4}204799$	
4	1032	178326	
254	$_{3}295480$	-26473	
4	787	23790	
258	296267	-2683	
4	787	2676	
$0_{3}262$	$_{4}297054$	-7	
· ·	16·		
	29721		
	16		
	29737		
	· ·		

From the given equation, by dividing by $x-0·543689$, we get $x^2+1·543689x+1·839287=0$ the roots of which are imaginary *, and are easily exhibited.

column. In this way the trial of $x=·3$, in the original equation, will stand as follows, the value of fx turning out to be $-291·36927$.

-70	1500	-58000	440000	-30000000 | 3
3	-201	3897	-162309	863073
-67	1299	-54103	277691	291·36927

It would be found in a similar manner, that for $x=·1$, $x=·2$, $x=·5$, and $x=·9$, the corresponding values of fx would be $-296·16569$, $-293·41088$, $-291·03125$, and $-300·44721$.

* It may be shown otherwise, and more easily, that the proposed equation can have no negative root. Thus, putting it under the form $(x^2+x+1)x-1=0$, we readily see that if x be negative, the quantity

Exer. 9.

Resolve the equation, $x^4+x^3+x^2+x-1=0$.

By Descartes's rule, this equation cannot have more than one positive root, nor more than three negative ones; and (§§ 187. and 188.) the roots must lie between 2 and -2. Their first figures are readily found to be 0 and -1; and the work for computing them is as follows: —

10	100	1000	-10000	0·5187902
5	75	875	9375	
15	175	1875	$-_{2}6250000$	
5	100	1375	3290301	
20	275	$_{2}3250000$	$-_{3}2959699$	
5	125	40301	2691024	
25	$_{2}40000$	3290301	$-_{4}268675$	
5	301	40603	237979	
$_{2}300$	40301	$_{3}3330904$	$-_{5}30696$	
1	302	3288·	30623	
301	40603	336378	-73	
1	303	3304	68	
302	$_{3}40906$	$_{4}339682$		
1	2·	29·		
303	411	33997		
1	2	29		
$_{3}304$	413	$_{5}34026$		
·	2	···		
	$0_{4}415$			
	· ·			

within the vinculum is positive. Its product, therefore, by x is negative, so that the first member can never be equal to zero. The same may be shown to be the case in every equation of the form $x^m+x^{m-1}+\ldots+x-1=0$, when m is an odd positive whole number.

1	1	1	−1	−1·2906488
−1	0	−1	0	
0	1	0	$-_2 10000$	
−1	1	−2	5856	
−1	2	$-_2 2000$	$-_3 41440000$	
−1	2	−928	41103981	
−2	$_2 400$	−2928	$-_4 336019$	
−1	64	−1064	310728	
$-_2 30$	464	$-_3 3992000$	−25291	
−2	68	−575109	20732	
−32	532	−4567109	−4559	
−2	72	−607347	4146	
−34	$_3 60400$	$-_4 5174456$	−413	
−2	3501	−43··	414	
−36	63901	−51788		
−2	3582	−43		
$-_3 380$	67483	−51831		
−9	3663	...		
−389	$0_4 71146$			
−9	. . .			
−398				
−9				
−407				
−9				
$-_4 416$				

The factors corresponding to the two roots which we have found, are $x-0·518790$ and $x+1·290649$. The product of these is

$$x^2+0·771859x-0·669576:$$

and by dividing the given equation by this, we get

$$x^2+0·228141x+1·493483=0,$$

the roots of which are imaginary.

Exer. 10.

Resolve the equation, $x^5+x^4+x^3+x^2+x-1=0$.

Here it is readily found in the usual way, that there is one positive root, and that it lies between 0 and 1; and its computation is as follows: —

10	100	1000	10000	-100000	0·50866049
5	75	875	9375	96875	
15	175	1875	19375	$-{}_2 3125000$	
5	100	1375	16250	2883856	
20	275	3250	${}_2 3562500$*	$-{}_3 241144$	
5	125	2000	4232·	219042	
25	400	${}_2 52500$	360482	$-{}_4 22102$	
5	150	4·	4264	21923	
30	${}_2 550$	529	${}_3 364746$	-179	
5	·	4	32·	146	
${}_2 35$		533	36507	-33	
·		4	32	32	
		537	${}_4 36539$		
		·	· · · ·		

For the reasons stated in the note to Exer. 8., p. 107., the other roots are imaginary.†

* For avoiding the use of too large numbers, only two ciphers are annexed here, and three in the succeeding column; while in the column immediately preceding, only one is added, and in the one before that one, none; and a figure is cut off from the number in the first column.

† The four equations contained in Exercises 7, 8, 9, and 10. are comprehended in a general one, which may be put under the form,

$$x+x^2+x^3+\dots\dots+x^n=1.$$

Now it is plain that, unless $n=1$, x cannot be 1 or greater than 1, as the first member would be greater than 1, the second. It is plain also, that if n, the number of terms, were infinite, the value of x would be $\frac{1}{2}$, as (Exam. 8. p. 112. ALG.) the sum of the infinite series, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c., is 1. The higher, accordingly, the value of n is, the more nearly does x approach to the extreme value, $\frac{1}{2}$.

We saw in the note to the solution of Exer. 8., that when n is odd,

Equations having Equal Roots.

(ALGEBRA, p. 194.)

Exer. 1.

Here $fx=x^3+x^2-x-1$, and therefore $f'x=3x^2+2x-1$. The greatest common measure of these, found by § 81. or § 82., is $x+1$. Hence there are two roots, each equal to -1. Then, by taking -2, the sum of these from -1, the coefficient of the second term of fx, with its sign changed, we find (according to § 176.) the remaining value of x to be 1.*

Exer. 2.

Here $fx=x^3-9x^2+27x-27$; and by finding $f'x$, and for simplicity dividing it by 3, we get x^2-6x+9. Then the

there can be no negative root; so that there can be such a root only when the equation is of the form,

$$x+x^2+x^3+\ldots.+x^{2m-1}+x^{2m}=1.$$

Now, in this case, x cannot be -2, nor between -2 and $-\infty$, as in either case, each term with an even index would exceed the term preceding it by 2 or upwards, and therefore the first member could not be equal to the second. Neither can x be between 0 and -1; as then the first and second terms taken together would give a negative amount, as would also the third and fourth, the fifth and sixth, &c.; and the entire amount of the first member could not be 1. The root, therefore, must lie between -1 and -2; and it is easy to see that it will approach the former more and more nearly, as the degree of the equation becomes higher: as it is only in this way that values of the several pairs of consecutive terms can be made so small, that the sums of those values shall amount to no more than 1.

* The given equation may be put under the form

$$x^2(x+1)-1(x+1)=0, \text{ or } (x^2-1)(x+1);$$

the three simple factors of which are $x-1$, $x+1$, and $x+1$, and therefore $x=1$, $x=-1$, and $x=-1^*$; so that the solution of this particular question, is thus obtained independently of the method given in the ALGEBRA.

greatest common measure of this and fx is found by either method, to be x^2-6x+9 or $(x-3)^2$. Hence the three roots are each equal to 3.

Exer. 3.

Here,

$$fx=x^4-2x^3-3x^2+4x+4, \text{ and } f'x=4x^3-6x^2-6x+4,$$

the greatest common measure of which is

$$x^2-x-2 \text{ or (§ 161.) } (x+1)(x-2).$$

Hence there are two roots, each $=-1$, and two others, each $=2$.

Exer. 4.

Since $fx=x^5-15x^3-10x^2+60x+72$, we have $f'x=5x^4-45x^2-20x+60$. Then the work for finding the common measure will stand as in the margin;

$$\begin{array}{rrrrrrl}
1 & 0 & -15 & -10 & 60 & 72 & \ldots\ldots(1.) \\
1 & 0 & -9 & -4 & 12 & & \ldots\ldots\ldots\ldots(2.) \\
\hline
 & & 6)6 & 6 & -48 & -72 & \\
\hline
 & & 1 & 1 & -8 & -12 & \ldots\ldots(3.)
\end{array}$$

line (2.) being obtained by dividing $f'x$ by 5. Line (3.) is then found from (1.) and (2.) by subtraction, and by dividing the remainder by 6. As this result is so simple, it is natural to suppose that it may measure (1.) and (2.); and by dividing each of them by it, it is found to succeed. We put, therefore, $x^3+x^2-8x-12=0$; and by means of § 189., we readily find the roots to be -2, -2, and 3. Hence, (§ 195.) the given equation has for roots -2, -2, -2, 3, and 3.

Exer. 5.

We have here $fx=x^5-6x^4+3x^3+46x^2-108x+72$, and consequently $f'x=5x^4-24x^3+9x^2+92x-108$. Then the operation for finding the greatest common measure, according to § 82., will be as follows :—

1	− 6	3	46	−108	72	(1.)
5	−24	9	92	−108		(2.)
5	−30	15	230	−540	360	(3.)
	6)6	− 6	−138	432	−360	
	1	− 1	− 23	72	− 60	(4.)
3	−18	9	138	−324	216	(5.)
	10	−48	18	184	−216	(6.)
3	− 8	−39	156	−140		(7.)
3	− 3	−69	216	−180		(8.)
	5)5	−30	60	− 40		
	1	− 6	12	− 8		(9.)
		5)5	− 35	80	− 60	
		1	− 7	16	− 12	(10.)
		1	− 4	4		(11.)

In this, (3.) is got from (1.) by multiplying by 5; and (4.) from (2.) and (3.) by subtraction, and by dividing by 6. Lines (5.) and (6.) are obtained from (1.) and (2.) by multiplying by 3 and 2; and (7.) from (5.) and (6.) by addition. We then get (8.) from (4.) by multiplying by 3.; and (9.) is derived from (7.) and (8.) by subtraction, and by dividing by 5. Line (10.) is obtained by taking (9.) from (4.) and by dividing the remainder by 5; and (11.) is the difference of (10.) and (9.). Now (11.) will be found by division, to be a measure of (1.) and (2.); and we put, therefore, $x^2-4x+4=0$, or, what is the same, $(x-2)^2=0$. Hence (§ 195.) the given equation has three roots each equal to 2: and by dividing it by the cube of $x-2$, we get $x^2-9=0$, and therefore $x=\pm 3$.

Reciprocal and Binomial Equations.

ALGEBRA, p. 199.)

Exer. 1.

In the solution of this exercise given in the margin, equation

(1.) is derived from the given equation by connecting the first and last terms, and those equally distant from them, and dividing by x^2; and from this, equation (2.) is obtained, by means of equation (5.), ALGEBRA, page 196. Equations (3.) and (4.) are derived from (2.) by obvious operations, and by ALGEBRA, § 152. Then (5.) and (7.) are obtained by substituting, successively, the values of y in equation (9.), ALGEBRA, page 196.; and from them we derive (6.) and (8.) by contraction.

$$3(x^2+x^{-2})+2(x+x^{-1})-34=0 \quad \ldots (1.)$$
$$3(y^2-2)+2y-34=0 \quad \ldots (2.)$$
$$3y^2+2y=40 \quad \ldots (3.)$$
$$y=\tfrac{10}{3} \text{ and } y=-4 \quad \ldots (4.)$$
$$x=\tfrac{1}{2}(\tfrac{10}{3}\pm\tfrac{8}{3}) \quad \ldots (5.)$$
$$x=3, \text{ and } x=\tfrac{1}{3}, \text{ or } x=3^{\pm1} \quad \ldots (6.)$$
$$x=\tfrac{1}{2}(-4\pm2\sqrt{3}) \quad \ldots (7.)$$
$$x=-2\pm\sqrt{3} \quad \ldots (8.)$$

Exer. 2.

Here (ALGEBRA, § 197.) one of the roots is -1. Dividing, therefore, by $x+1$, putting the quotient $=0$, connecting the first and last terms, the terms next them, &c., and dividing by x^2, we get equation (1.); which by (5.) and (3.) Exam. 1., gives (2.); and from this (3.) is derived by easy operations. The resolution of this gives (4.) and (5.); which become (6.) by substituting its value for y. The rest of the work consists in multiplying in the first of the equations marked (6.) by $2x$, and in the second by x, and resolving the equations so obtained.

$$2(x^2+x^{-2})-3(x+x^{-1})-1=0 \quad \ldots (1.)$$
$$2(y^2-2)-3y-1=0 \quad \ldots (2.)$$
$$2y^2-3y=5 \quad \ldots (3.)$$
$$y=\frac{3\pm\sqrt{(9+40)}}{4} \quad \ldots (4.)$$
$$y=\tfrac{5}{2}, \text{ and } y=-1 \quad \ldots (5.)$$
$$\left.\begin{array}{l} x+x^{-1}=\tfrac{5}{2}, \\ \text{and } x+x^{-1}=-1 \end{array}\right\} \quad \ldots (6.)$$
$$x=\frac{5\pm\sqrt{(25-16)}}{4}$$
$$=2, \text{ and } x=\tfrac{1}{2} \quad \ldots (7.)$$
$$x^2+x=-1 \quad \ldots (8.)$$
$$\left.\begin{array}{l} x=\tfrac{1}{2}\{-1\pm\sqrt{(1-4)}\} \\ \ =\tfrac{1}{2}(-1\pm\sqrt{-3}) \end{array}\right\} \quad \ldots (9.)$$

Exer. 3.

In this exercise, according to § 198., two of the roots are 1 and -1; and equation (1.) is obtained from the given equation by dividing by x^2-1. From this (2.) is got by the usual method of connecting the terms, and by dividing by x^2; and (3.) is the same as (2.) modified according to (3.) and (5.) in Exam. 1., p. 196. From (3.) we derive (4.) by obvious operations; and (5.) is got by the resolution of (4.). Equations (6.) and (8.) are obtained by giving to y, or its equal $x+x^{-1}$, the two values found in (5.); and (7.) and (9.) are got from these by multiplying by $4x$, and by resolving the resulting equations.

$$16x^4-64x^3+15x^2-64x+16=0 \quad \ldots (1.)$$
$$16(x^2+x^{-2})-64(x+x^{-1})+15=0 \quad \ldots (2.)$$
$$16(y^2-2)-64y+15 \quad \ldots (3.)$$
$$16y^2-64y=17 \quad \ldots (4.)$$
$$y=\tfrac{17}{4}, \text{ and } y=-\tfrac{1}{4} \quad \ldots (5.)$$
$$x+x^{-1}=\tfrac{17}{4} \quad \ldots (6.)$$
$$x=4, \text{ and } x=\tfrac{1}{4} \quad \ldots (7.)$$
$$x+x^{-1}=-\tfrac{1}{4} \quad \ldots (8.)$$
$$x=\tfrac{1}{8}(-1\pm\sqrt{-63}) \quad \ldots (9.)$$

Exer. 4.

Here, as in the preceding exercise, two of the roots are 1 and -1, and we get equation (1.) from the original equation, by dividing by x^2-1. Equation (2.) is obtained from (1.) by connecting the terms in the usual way and dividing by x^3; and (3.) is got from (2.) by equation (4.), Exam. 3., p. 197., and (3.), p. 196. From this (4.) is got by contraction, &c., and (5.) by extracting the cube root and dividing by 2. Equation (6.) is the same as (5.); and (7.) is obtained from it by multiplying by $2x$ and transposing. Lastly, (8.) is found by the resolution of (7.).

$$8x^6+24x^4-125x^3+24x^2+8=0 \quad \ldots (1.)$$
$$8(x^3+x^{-3})+24(x+x^{-1})-125=0 \quad \ldots (2.)$$
$$8(y^3-3y)+24y-125=0 \quad \ldots (3.)$$
$$8y^3=125 \quad \ldots (4.)$$
$$y=\tfrac{5}{2}\sqrt[3]{1} \quad \ldots (5.)$$
$$x+x^{-1}=\tfrac{5}{2}\sqrt[3]{1} \quad \ldots (6.)$$
$$2x^2-5\sqrt[3]{1}\cdot x=-2 \quad \ldots (7.)$$
$$x=\tfrac{1}{4}\{5\sqrt[3]{1}\pm\sqrt{[25(\sqrt[3]{1})^2-16]}\} \quad \ldots (8.)$$

Exer. 5.

Here we have $1024=1024\times 1$, and, therefore, $1024^{\frac{1}{5}}=4\times 1^{\frac{1}{5}}$. Then one of the fifth roots of 1 being 1, we have 4 as one of the fifth roots of 1024. Let now x be put to denote a fifth root of 1, and we shall have $x^5=1$ or $x^5-1=0$. Dividing this by $x-1$, we get equation (1.), and from it (2.) is obtained in the usual way, by connecting the first and last terms, and dividing by x^2. Equation (3.) is then had by means of (5.) and (3.), ALGEBRA, p. 196.; and (4.), and which is the same, (5.), by the resolution of this. In the last place, (6.) is found by multiplying by $2x$ and transposing, and (7.) by resolving (6.).

$$x^4+x^3+x^2+x+1=0 \quad \ldots\ldots\ldots \quad (1.)$$

$$x^2+x^{-2}+x+x^{-1}+1=0 \quad \ldots\ldots \quad (2.)$$

$$y^2-2+y+1=0 \quad \ldots\ldots\ldots\ldots \quad (3.)$$

$$y=\tfrac{1}{2}(-1\pm\sqrt{5}) \quad \ldots\ldots\ldots\ldots \quad (4.)$$

$$x+x^{-1}=\tfrac{1}{2}(-1\pm\sqrt{5}) \quad \ldots\ldots\ldots \quad (5.)$$

$$2x^2+(1\mp\sqrt{5})x=-2 \quad \ldots\ldots\ldots \quad (6.)$$

$$x=\frac{-1\pm\sqrt{5}\pm\sqrt{(-10\pm2\sqrt{5})}}{4} \quad . \quad (7.)$$

Exer. 6.

In this exercise we have (§ 198.) $x=1$ and $x=-1$; and, by dividing by x^2-1, we get $x^2+1=0$; whence $x=\pm\sqrt{-1}$.

INDETERMINATE COEFFICIENTS.

(ALGEBRA, p. 202.)

Exer. 1.

Here it is easy to see that the first term of the answer must be 1; and as the divisor contains successive powers of x, it is natural to suppose that the required quotient will be composed of such powers. Hence we assume

$$\frac{1}{1-x+x^2}=1+A_1x+A_2x^2+A_3x^3+A_4x^4+\text{ \&c. } \ldots \quad (1.)$$

Hence, by multiplying by $1-x+x^2$, we get

$$\left.\begin{array}{r|r|r|r|r} 1=1+A_1 & x+A_2 & x^2+A_3 & x^3+A_4 & x^4+\&c. \\ -1 & -A_1 & -A_2 & -A_3 & -\&c. \\ & +1 & +A_1 & +A_2 & +\&c. \end{array}\right\} \quad . . (2.)$$

We have accordingly,

$$A_1-1=0, \qquad A_2-A_1+1-0,$$
$$A_3-A^2+A_1=0, \qquad A_4-A_3+A_2=0,$$
$$A_5-A_4+A_3=0, \text{ \&c.}:$$

and hence,

$$A_1=1, \qquad A_2=A_1-1=0,$$
$$A_3=A_2-A_1=0-1=-1, \qquad A_4=A_3-A_2=-1,$$
$$A_5=A_4-A_3=-1+1=0, \text{ \&c.};$$

and the substitution of these in (1.) gives the answer in its first form. The second form may be obtained by dividing the first by $1+x$, and then indicating the multiplication of the quotient by the same.*

Exer. 2.

Let $1+A_1x+A_2x^2+A_3x^3+$ &c. be assumed as equivalent to the given fraction. Then by multiplying by $1+x+x^2+x^3$, we get

$$\begin{array}{r|r|r|r|r} 1=1+A_1 & x+A_2 & x^2+A_3 & x^3+A_4 & x^4+\&c. \\ +1 & +A_1 & +A_2 & +A_3 & +\&c. \\ & +1 & +A_1 & +A_2 & +\&c. \\ & & +1 & +A_1 & +\&c. \end{array}$$

Hence $A_1+1=0, \qquad A_2+A_1+1=0,$
$A_3+A_2+A_1+1=0, \qquad A_4+A_3+A_2+A_1=0,$ &c.

The first of these gives $A_1=-1$, the second $A_2=0$, the third

* This exercise may also be neatly and easily solved by multiplying the numerator and denominator of the given fraction by $1+x$, as by this means it is transformed into $(1+x)\dfrac{1}{1+x^3}$. Then, to evolve the second factor in a series, assume it equal to $1+A_3x^3+A_6x^6+$ &c.; and by multiplying by $1+x^3$, equating the coefficients, &c., the series will be found to be $1-x^3+x^6-x^9+$ &c.; and the answer is the product of this by $1+x$.

$A_3=0$, the fourth $A_4=1$, &c.; and the first form of the answer is found by substituting these in the assumed series. The second form may be obtained from the first by dividing by $1-x$, and indicating the multiplication of the quotient by the same.*

Exer. 3.

In this exercise we might assume $1+A_1x+A_2x^2+$ &c., as the root; and by squaring and equalling the coefficients, &c., we should get the values of A_1, A_2, &c. By a little consideration, however, it is easy to see that the root will contain none of the odd powers of x, since, if there were any such, its square must contain odd powers, which, however, do not exist in the proposed quantity, $1+x^2$. Let us assume, therefore,

$$\sqrt{(1+x^2)}=1+A_2x^2+A_4x^4+A_6x^6+\text{ \&c.} \ldots\ldots (1.)$$

Then, by squaring, we get

$$1+x^2=1+\begin{array}{c|c|c|c} A_2 & x^2+A_4 & x^4+A_6 & x^6+\text{\&c.} \\ +A_2 & +A_2^2 & +A_2A_4 & +\text{\&c.} \\ & +A_4 & +A_2A_4 & +\text{\&c.} \\ & & +A_6 & +\text{\&c.} \end{array}\Bigg\} \ldots (2.)$$

Hence,

$$2A_2=1, \qquad 2A_4+A_2^2=0, \qquad 2A_6+2A_2A_4=0, \text{ \&c.};$$

and consequently

$$A_2=\tfrac{1}{2}, \qquad A_4=-\tfrac{1}{2}A_2^2=-\frac{1}{2\cdot 4}$$

$$A_6=-A_2A_4=\frac{1}{2\cdot 2\cdot 4}=\frac{1\cdot 3}{2\cdot 4\cdot 6}, \text{ \&c.};$$

the substitution of which in (1.) gives the answer.†

* By multiplying the terms of the proposed fraction by $1-x$, we reduce it to the form, $(1-x)\dfrac{1}{1-x^4}$. Then the series equivalent to the second factor is readily obtained by assuming it equal to $1+A_4x^4+A_8x^8+$ &c., by multiplying by $1-x^4$, &c.

† The result required in this exercise is easily found from the answer to the example in the ALGEBRA, page 201., by taking $a=1$, and changing x into x^2.

Binomial Theorem.

(ALGEBRA, p. 208.)

Exer. 1.

Here $n=5$ and $\frac{1}{2}(n-1)=2$; and, by § 211., the first term of the answer will be a^5. From this the second is derived by multiplying by 5 and by x, and dividing by a: it is, therefore, $5a^4x$, and by multiplying this by 2 and x, and dividing by a, we find the third to be $10a^3x^2$. Then, according to the note in the ALGEBRA, page 206., there will be six terms in the whole development, and the coefficients of the three still to be found will be 10, 5, and 1, the same as those obtained already, taken in a reversed order. The required power, therefore, is

$$a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5.$$

Exer. 2.

In this exercise the coefficients will be the same as in the last, the index being the same. The first term will be a^{10} (the fifth power of a^2); and from this the next term without its coefficient is found, by multiplying by $-x^2$ and dividing by a^2, to be $-a^8x^2$; from this again, by a like process, the third, without its coefficient, is found to be a^6x^4; and so the other terms may be derived. The law of formation, however, is evident without the formal performance of such operations, the index of a being continually diminished by 2, and that of x increased by the same, and the terms being alternately positive and negative in consequence of the multiplications by $-x^2$.*

Exer. 3.

Here, following the method adopted in the solution of Exam. 2., ALGEBRA, page 207., since the index is $\frac{1}{2}$, we have $p=1$ and

* The answer to this exercise may be derived at once from that of the last, simply by changing a into a^2, and x into $-x^2$.

$q=2$; and therefore the multipliers $\frac{1}{2}(n-1)$, $\frac{1}{3}(n-2)$, &c. are found to be

$$-\tfrac{1}{4},\ -\tfrac{3}{6},\ -\tfrac{5}{8},\ -\tfrac{7}{10},\ \&c.$$

Then, by § 211., we have

$$(a+x)^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{1}{2}a^{\frac{1}{2}}\cdot\frac{x}{a}-\frac{1}{4}\cdot\frac{1}{2}a^{\frac{1}{2}}\cdot\frac{x}{a}\cdot\frac{x}{a}+\frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}a^{\frac{1}{2}}\cdot\frac{x}{a}\cdot\frac{x}{a}\cdot\frac{x}{a}$$

$$-\frac{5}{8}\cdot\frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}a^{\frac{1}{2}}\cdot\frac{x}{a}\cdot\frac{x}{a}\cdot\frac{x}{a}\cdot\frac{x}{a}+\ \&c.$$

This, by dividing all the terms of the second member by $a^{\frac{1}{2}}$ and indicating the multiplication of the quotient by the same, and by some obvious modifications, will take the form $(a+x)^{\frac{1}{2}}=$

$$a^{\frac{1}{2}}\left(1+\frac{1}{2}\cdot\frac{x}{a}-\frac{1}{2\cdot 4}\cdot\frac{x^2}{a^2}+\frac{1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{x^3}{a^3}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{x^4}{a^4}+\ \&c.\right);$$

which, except a slight change of form, is the same as the answer in the ALGEBRA.

Exer. 4.

Between this exercise and the last, the sole difference is, that the sign of x is contrary. The answer of this exercise, therefore, will be obtained from that of the foregoing, merely by changing (§ 118.) the signs of the terms which contain the odd powers of x as factors.*

Exer. 5.

Here we have $p=1$ and $q=3$, and the multipliers,

$$\tfrac{1}{3}(n-1),\ \tfrac{1}{3}(n-2),\ \&c.,$$

are readily found to be

$$-\tfrac{2}{6},\ -\tfrac{5}{9},\ -\tfrac{8}{12},\ \&c.$$

* It will be useful for the student to find the answer to this exercise independently of that of the last; and from it, when so found, to derive the answer to the last by changing the signs of the odd powers of x.

The first term of the development, also, according to § 211., is $a^{\frac{1}{3}}$; and the second, third, &c., without their coefficients are

$$\frac{a^{\frac{1}{3}}x}{a},\ \frac{a^{\frac{1}{3}}x^2}{a^2},\ \frac{a^{\frac{1}{3}}x^3}{a^3},\ \&c.:$$

and the coefficient of the second term is $\frac{1}{3}$; that of the third, $\frac{1}{3}\times-\frac{2}{6}$, or $-\frac{2}{3\cdot6}$; of the fourth, $-\frac{2}{3\cdot6}\times-\frac{5}{9}$ or $\frac{2\cdot5}{3\cdot6\cdot9}$, &c. Hence the answer will be obtained by applying these coefficients to their proper terms, and the result will be conveniently modified by dividing all the terms by $a^{\frac{1}{3}}$, and indicating the multiplication of the quotient by the same.

Exer. 6.

In solving this question, the process will be exactly the same as in the preceding, except that a^3 and x^3 must be used instead of a and x. The answer, in fact, will be derived from that of the last by changing a into a^3, and x into x^3.

Exer. 7.

Between this and the last, the sole difference is, that we have here $-x^3$ instead of x^3, and consequently, by extraction, $-x$ instead of x; and therefore the answer will be obtained from that of the last (ALGEBRA, § 118.), by changing the signs of the terms containing odd powers of x.*

Exer. 8.

Here the quantity to be expanded is $(a+x)^{-1}$, so that $n=-1$; and, consequently,

$$\tfrac{1}{2}(n-1)=-1,\ \tfrac{1}{3}(n-2)=-1,\ \tfrac{1}{4}(n-3)=-1,\ \&c.$$

The terms, also, without the coefficients, are (§ 211.) a^{-1}, $a^{-2}x$,

* The student, for his improvement, should solve this question and the preceding independently of one another, and of Exam. 5.; and he will feel no difficulty in effecting this after having made himself acquainted with the solution of the 5th. If he do this, he will find, that from the solution of any one of the three questions, the solutions of the other two may be derived.

$a^{-3}x^2$, &c.: and therefore the answer given in the ALGEBRA is found by means of the formula in § 211., and by taking the quantities having negative indices to the denominators.

By taking $x+a$ as denominator, we should get, as another answer,

$$\frac{1}{x}-\frac{a}{x^2}+\frac{a^2}{x^3}-\frac{a^3}{x^4}+\text{\&c.};$$

and the like may be done in many other cases.

Exer. 9.

In expanding this quantity, $(a+x)^{-2}$, we have $n=-2$; and, consequently,

$$\tfrac{1}{2}(n-1)=-\tfrac{3}{2},\ \tfrac{1}{3}(n-2)=-\tfrac{4}{3},\ \tfrac{1}{4}(n-3)=-\tfrac{5}{4},\ \text{\&c.}$$

Also (ALGEBRA, § 211.), the terms without the coefficients are

$$a^{-2},\ a^{-3}x,\ a^{-4}x^2,\ \text{\&c.}$$

The formula in § 211. gives us, therefore,

$$(a+x)^{-2}=a^{-2}-2a^{-3}x+3a^{-4}x^2-4a^{-5}x^3+5a^{-6}x^4-\ \text{\&c.},$$

the answer in the ALGEBRA, when the quantities with negative indices are taken to the denominators.

Exer. 10.

Here, since the quantity to be expanded is $a^3(a^3-x^3)^{-\frac{2}{3}}$, the latter factor is to be developed by the binomial theorem, and the result is to be multiplied by a^3. In expanding $(a^3-x^3)^{-\frac{2}{3}}$, we have $p=-2$ and $q=3$ (or $p=2$ and $q=-3$); and, therefore,

$$\tfrac{1}{2}(n-1)=-\tfrac{5}{6},\ \tfrac{1}{3}(n-2)=-\tfrac{8}{9},\ \tfrac{1}{4}(n-3)=-\tfrac{11}{12},\ \text{\&c.}$$

Also (§ 211.), the terms without the coefficients are

$$(a^3)^{-\frac{2}{3}} \text{ or } a^{-2};\ -a^{-2}.a^{-3}.x^3 \text{ or } -a^{-5}x^3;\ -a^{-5}.a^{-3}.-x^3= a^{-8}x^6,\ \text{\&c.}$$

As to the coefficients of the several terms, the first (§ 211.) is 1; the second, $-\frac{2}{3}$; the third, $-\frac{2}{3}\times-\frac{5}{6}=\frac{2\cdot5}{3\cdot6}$; the fourth, $\frac{2\cdot5}{3\cdot6}\times-\frac{8}{9}=-\frac{2\cdot5\cdot8}{3\cdot6\cdot9}$; &c. Substituting the values thus obtained in the series in § 211., multiplying by a^3, and taking the quan-

tities with negative indices to the denominators, we readily obtain the answer given in the ALGEBRA.

Exer. 11.

Here $(a-x)(a+x)^{-\frac{1}{3}}$ is the quantity to be expanded. The required result, therefore, will be obtained by expending $(a+x)^{-\frac{1}{3}}$ by the binomial theorem, and multiplying the series so obtained by $a-x$. In expanding $(a+x)^{-\frac{1}{3}}$ by means of § 211., we have as the successive terms without the coefficients,

$$a^{-\frac{1}{3}}, \qquad a^{-\frac{4}{3}}x, \qquad a^{-\frac{7}{3}}x^2, \text{ \&c.};$$

and as the successive multipliers, $\frac{1}{2}(n-1)$, $\frac{1}{3}(n-2)$, &c.,

$$-\tfrac{4}{6}, \; -\tfrac{7}{9}, \; -\tfrac{10}{12}, \text{ \&c.}$$

Hence, by § 211., we get the answer in the ALGEBRA, after multiplying by $a^{\frac{1}{3}}$ and by $a-x$, and by indicating the division by $a^{\frac{1}{3}}$.

CONTINUED FRACTIONS.

(ALGEBRA, p. 219.)

Exer. 1.

The work for finding the several quotients will be as follows: (See the Author's Arithmetic, page 81.)

```
965)351(0
     351)965(2
         702
         263)351(1
             263
              88)263(2
                 176
                  87)88(1
                     87
                      1)87(87
                        0
```

Then the work for finding the converging fractions will stand thus:

$$0,\ 2,\ 1,\ 2,\ 1,\ 87,$$
$$\frac{0}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{3}{8},\ \frac{4}{11},\ \frac{351}{965}.$$

Exer. 2.

In this exercise the quotients are found thus:

```
251)764(3
    753
    ---
     11)251(22
         22
         --
          31
          22
          --
           9)11(1
              9
              -
              2)9(4
                8
                -
                1)2(2
                  2
                  -
                  0
```

Then the process for finding the converging fractions will be as follows:

$$3,\ 22,\ 1,\ 4,\ 2,$$
$$\frac{3}{1},\ \frac{67}{22},\ \frac{70}{23},\ \frac{347}{114},\ \frac{764}{251}.$$

Exer. 3.

```
1769)5537(3
     5307
     ----
      230)1769(7
          1610
          ----
           159)230(1
               159
               ---
                71)159(2
                   142
                   ---
                    17)71(4
                       68
                       --
                        3)17(5
                          15
                          --
                           2)3(1
                             2
                             -
                             1)2(2
                               2
                               -
                               0
```

$$3,\quad 7,\quad 1,\quad 2,\quad 4,\quad 5,\quad 1,\quad 2,$$
$$\frac{3}{1},\ \frac{22}{7},\ \frac{25}{8},\ \frac{72}{23},\ \frac{313}{100},\ \frac{1637}{523},\ \frac{1950}{623},\ \frac{5537}{1769}.$$

Exer. 4.*

```
121)196(1
    121
    ---
     75)121(1
         75
         --
         46)75(1
            46
            --
            29)46(1
               29
               --
               17)29(1
                  17
                  --
                  12)17(1
                     12
                     --
                      5)12(2
                        10
                        --
                         2)5(2
                           4
                           -
                           1)2(2
                             2
                             -
                             0
```

$$1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 2,\quad 2,\quad 2,$$
$$\frac{1}{1},\ \frac{2}{1},\ \frac{3}{2},\ \frac{5}{3},\ \frac{8}{5},\ \frac{13}{8},\ \frac{34}{21},\ \frac{81}{50},\ \frac{196}{121}.$$

* The ratio of 196 to 121 is that of the old Irish "Statute acre" to the imperial acre, 121 of the former being equivalent to 196 of the latter.

Exer. 5.

$$\sqrt{2}=1+\frac{1}{x_1}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (1.)$$

$$x_1=\frac{1}{\sqrt{2}-1}=\sqrt{2}+1=2+\frac{1}{x_2}\ \ldots\ (2.)$$

$$x_2=\frac{1}{\sqrt{2}-1}=x_1\ \ldots\ldots\ldots\ldots\ldots\ (3.)$$

$$\sqrt{2}=1+\frac{1}{2+}\frac{1}{2+}\ \&c.$$

Here, by comparing lines (2.) and (3.), we see that the values of x_1, x_2, x_3, &c., would all be the same, so that each of the denominators must be 2. The work for finding the converging fractions is as follows: —

$$1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ \&c.$$
$$\tfrac{1}{1},\ \tfrac{3}{2},\ \tfrac{7}{5},\ \tfrac{17}{12},\ \tfrac{41}{29},\ \tfrac{99}{70},\ \tfrac{239}{169},\ \&c.$$

Exer. 6.

$$\sqrt{28}=5+\frac{1}{x_1}$$

$$x_1=\frac{1}{\sqrt{28}-5}=\frac{\sqrt{28}+5}{3}=3+\frac{1}{x_2}$$

$$x_2=\frac{3}{\sqrt{28}-4}=\frac{\sqrt{28}+4}{4}=2+\frac{1}{x_3}$$

$$x_3=\frac{4}{\sqrt{28}-4}=\frac{\sqrt{28}+4}{3}=3+\frac{1}{x_4}$$

$$x_4=\frac{3}{\sqrt{28}-5}=\sqrt{28}+5=10+\frac{1}{x_5}$$

$$x_5=\frac{1}{\sqrt{28}-5}=x_1.$$

Hence, the denominators of the fractions are 3, 2, 3, 10, repeated without limit. The converging fractions are found as follows :—

$$5,\ 3,\ 2,\ 3,\ 10,\ 3,\ \&c.$$
$$\tfrac{5}{1},\ \tfrac{16}{3},\ \tfrac{37}{7}.\ \tfrac{127}{24},\ \tfrac{1307}{247},\ \tfrac{4048}{765},\ \&c.$$

Exer. 7.

$$\sqrt{45}=6+\frac{1}{x_1}.$$

$$x_1=\frac{1}{\sqrt{45}-6}=\frac{\sqrt{45}+6}{9}=1+\frac{1}{x_2}$$

$$x_2=\frac{9}{\sqrt{45}-3}=\frac{\sqrt{45}+3}{4}=2+\frac{1}{x_3}$$

$$x_3=\frac{4}{\sqrt{45}-5}=\frac{\sqrt{45}+5}{5}=2+\frac{1}{x_4}$$

$$x_4=\frac{5}{\sqrt{45}-5}=\frac{\sqrt{45}+5}{4}=2+\frac{1}{x_5}$$

$$x_5=\frac{4}{\sqrt{45}-3}=\frac{\sqrt{45}+3}{9}=1+\frac{1}{x_6}$$

$$x_6=\frac{9}{\sqrt{45}-6}=\sqrt{45}+6=12+\frac{1}{x_7}$$

$$x_7=\frac{1}{\sqrt{45}-6}=x_1$$

Hence, the succeeding quotients will be 1, 2, 2, 2, 1, 12, repeated without end. The work for finding the converging fractions will stand thus: —

$$6,\ 1,\ 2,\ 2,\ 2,\ 1,\ 12,\ \&c.$$

$$\tfrac{6}{1},\ \tfrac{7}{1},\ \tfrac{20}{3},\ \tfrac{47}{7},\ \tfrac{114}{17},\ \tfrac{161}{24},\ \tfrac{2046}{305},\ \&c.$$

Exer. 8.

$$\sqrt{52}=7+\frac{1}{x_1}$$

$$x_1=\frac{1}{\sqrt{52}-7}=\frac{\sqrt{52}+7}{3}=4+\frac{1}{x_2}$$

$$x_2=\frac{3}{\sqrt{52}-5}=\frac{\sqrt{52}+5}{9}=1+\frac{1}{x_3}$$

$$x_3=\frac{9}{\sqrt{52}-4}=\frac{\sqrt{52}+4}{4}=2+\frac{1}{x_4}$$

$$x_4=\frac{4}{\sqrt{52}-4}=\frac{\sqrt{52}+4}{9}=1+\frac{1}{x_5}$$

$$x_5=\frac{9}{\sqrt{52}-5}=\frac{\sqrt{52}+5}{3}=4+\frac{1}{x_6}$$

$$x_6=\frac{3}{\sqrt{52}-7}=\sqrt{52}+7=14+\frac{1}{x_7}$$

$$x_7=\frac{1}{\sqrt{52}-7}=x_1.$$

The periodical quotients, therefore, are 4, 1, 2, 1, 4, and 14; and the work for finding the converging fractions will be as follows: —

$$7,\quad 4,\quad 1,\quad 2,\quad 1,\quad 4,\quad 14,\ \&c.$$

$$\tfrac{7}{1},\ \tfrac{29}{4},\ \tfrac{36}{5},\ \tfrac{101}{14},\ \tfrac{137}{19},\ \tfrac{649}{90},\ \tfrac{9223}{1279},\ \&c.$$

Exer. 9.

$$\sqrt{53}=7+\frac{1}{x_1}$$

$$x_1=\frac{1}{\sqrt{53}-7}=\frac{\sqrt{53}+7}{4}=3+\frac{1}{x_2}$$

$$x_2=\frac{4}{\sqrt{53}-5}=\frac{\sqrt{53}+5}{7}=1+\frac{1}{x_3}$$

$$x_3=\frac{7}{\sqrt{53}-2}=\frac{\sqrt{53}+2}{7}=1+\frac{1}{x_4}$$

$$x_4=\frac{7}{\sqrt{53}-5}=\frac{\sqrt{53}+5}{4}=3+\frac{1}{x_5}$$

$$x_5=\frac{4}{\sqrt{53}-7}=\sqrt{53}+7=14+\frac{1}{x_6}$$

$$x_6=\frac{1}{\sqrt{53}-7}=x_1.$$

The quotients, therefore, are 3, 1, 1, 3, 14, continually repeated. The process for finding the converging fractions will stand as follows: —

$$\begin{array}{cccccc c} 7, & 3, & 1, & 1, & 3, & 14, & \&c. \\ \frac{7}{1}, & \frac{22}{3}, & \frac{29}{4}, & \frac{51}{7}, & \frac{182}{25}, & \frac{2599}{357}, & \&c. \end{array}$$

Exer. 10.

In solving this exercise, the first quotient and remainder are found to be 4 and 2684; the second, 7 and 2141; the third, 1 and 543; the fourth, 3 and 512; the fifth, 1 and 31; the sixth, 16 and 16; the seventh, 1 and 15; the eighth, 1 and 1; and the ninth, 15 and 0. Then the process for finding the converging fractions will stand as follows: —

$$\begin{array}{ccccccccc} 4, & 7, & 1, & 3, & 1, & 16, & 1, & 1, & 15, \\ \frac{4}{1}, & \frac{29}{7}, & \frac{33}{8}, & \frac{128}{31}, & \frac{161}{39}, & \frac{2704}{655}, & \frac{2865}{694}, & \frac{5569}{1349}, & \frac{86400}{20929}. \end{array}$$

Exer. 11.

By reducing 365 days to seconds (by multiplying successively by 24, 60, and 60), we get 31,536,000; and adding to this 20929, we get 31,556,929, the number of seconds in the civil or solar year.* We find also, by successive multiplications by 24, 60, and 60, that the number of seconds in the mean synodical month is 2,551,443. Then, by the usual processes in division, we find the successive quotients and remainders to be 12 and 939613; 2 and 672217; 1 and 267396; 2 and 137425; 1 and 129971; 1 and 7454; 17 and 3253; 2 and 948; 3 and 409; 2 and 130; 3 and 19; 6 and 16; 1 and 3; 5 and 1; and 3 and 0. The converging fractions will be found as follows: —

$$\begin{array}{ccccccccccc} 12, & 2, & 1, & 2, & 1, & 1, & 17, & 2, & 3, & 2, & 3, \\ \frac{12}{1}, & \frac{25}{2}, & \frac{37}{3}, & \frac{99}{8}, & \frac{136}{11}, & \frac{235}{19}, & \frac{4131}{334}, & \frac{8497}{687}, & \frac{29622}{2395}, & \frac{67741}{5477}, & \frac{232845}{18826}, \end{array}$$

$$\begin{array}{cccc} 6, & 1, & 5, & 3, \\ \frac{1464811}{118433}, & \frac{1697656}{137259}, & \frac{9953091}{804728}, & \frac{31556929}{2551443} \end{array}$$

* The same would, of course, be found by reducing 365d. 5h. 48m. 49s. to seconds.

Exer. 12.

Here we may assume

$$x=3+\frac{1}{2+\frac{1}{1+\frac{1}{y}}};\quad \text{where } y=4+\frac{1}{2+\frac{1}{3+\frac{1}{y}}}.$$

Then, by the usual process, as in the margin, we find $x=\dfrac{10y+7}{3y+2}$, and $y=\dfrac{31y+9}{7y+2}$. From the first of these, by multiplying by the denominator, &c., we get $y=\dfrac{7-2x}{3x-10}$. Then, by substituting this for y in the second of the preceding equations, multiplying the numerator and denominator of the second member by $3x-10$, contracting, freeing the result of fractions, &c., we get the answer.

$$3,\quad 2,\quad 1,\quad y,$$
$$\frac{3}{1},\ \frac{7}{2},\ \frac{10}{3},\ \frac{10y+7}{3y+2}$$

$$4,\quad 2,\quad 3,\quad y,$$
$$\frac{4}{1},\ \frac{9}{2},\ \frac{31}{7},\ \frac{31y+9}{7y+2}$$

INDETERMINATE ANALYSIS.

(Algebra, p. 227.)

Exer. 1.

The work in this process is so simple as to require no explanation. To get positive values for y, it is plain that no positive value can be assigned to v; and that to have x positive, as well as y, v cannot be less than -3. This appears from the following table: —

$$2x+3y=25$$
$$x=\frac{25-3y}{2}=12-y+\frac{1-y}{2}$$
$$\frac{1-y}{2}=v$$
$$y=1-2v$$
$$x=11+3v$$

v	1,	0,	-1,	-2,	-3,	-4,	&c.
x	14,	11,	8,	5,	2,	-1,	&c.
y	-1,	1,	3,	5,	7,	9,	&c.

To solve this according to the method pointed out in the ALGEBRA, § 232., we have $a=2$, $b=3$, and $c=25$; and the quotients obtained by means of a and b are 0, 1, and 2. Also p and q, found as in the margin, are each equal to 1; so that $x=-25+3v$, and $y=25-2v$, the lower signs being used in the expressions found for x and y, in § 232., because p and q occupy an even place in the work in the margin. By changing in these v into $v+8$, we get $x=3v-1$, and $y=9-2v$. Also, by changing v into $v+12$, we get the same expressions for x and y that were found by the first method.

$$0,\quad 1,\quad 2;$$
$$\frac{0}{1},\ \frac{1}{1},\ \frac{2}{3}.$$

Exer. 2.

$$5x+7y=52.$$

$$x=\frac{52-7y}{5}=10-y+\frac{2-2y}{5}.$$

$$\frac{2-2y}{5}=v_1.$$

$$y=\frac{2-5v_1}{2}=1-2v_1-\frac{v_1}{2}.$$

$\frac{v_1}{2}=v$, and therefore $v_1=2v$.

$y=1-5v$, and $x=9+7v$.

v	1,	0,	−1,	−2,	&c.
x	16,	9,	2,	−5,	&c.
y	− 4,	1,	6,	11,	&c.

In solving this according to § 232., we have $a=5$, $b=7$, and $c=52$; and, as in the margin, we find $p=2$, and $q=3$: and thence we get $x=156+7v$, and $y=-104-5v$. By changing v into $v-22$, we obtain the simpler expressions, $x=2+7v$, and $y=6-5v$. Also by changing v into $v-21$, we find $x=9+7v$, and $y=1-5v$, the same as by the first method.

$$0,\ 1,\ 2,\ 2;$$
$$\frac{0}{1},\ \frac{1}{1},\ \frac{2}{3},\ \frac{5}{7}.$$

Exer. 3.

$$4x + 13y = 229.$$

$$x = \frac{229 - 13y}{4} = 57 - 3y + \frac{1-y}{4}.$$

$$\frac{1-y}{4} = v;\text{ and therefore } y = 1 - 4v,$$

and $x = 54 + 13v$.

The following table exhibits the values of x and y which merit consideration:

v	1,	0,	$-$ 1,	$-$ 2,	$-$ 3,	$-$ 4,	$-$ 5,	&c.
x	67,	54,	41,	28,	15,	2,	$-$11,	&c.
y	$-$ 3,	1,	5,	9,	13,	17,	21,	&c.

In employing the second method we get $p=1$, $q=3$, $x=-687+13v$, and $y=229-4v$. Then, by changing v first into $53+v$, and again into $57+v$, we get in the first place $x=2+13v$, and $y=17-4v$, and in the second $x=54+13v$, and $y=1-4v$; the latter two values agreeing with what was found by the first method.

Exer. 4.

$$3x + 5y = 7.$$

$$x = \frac{7-5y}{3} = \frac{6+1-6y+y}{3} = 2 - 2y + \frac{1+y}{3}.$$

Hence,

$$\frac{1+y}{3} = v,\text{ and therefore } y = 3v - 1,$$

and $x = 4 - 5v$.

v	0,	1,	2,	&c.
x	4,	$-$1,	$-$6,	&c.
y	$-$1,	2,	5,	&c.

In employing § 232., we get $p=1$, $q=2$, $x=14+5v$, and $y=-7-3v$. Simpler expressions for x and y may be found either by changing v into $-3+v$ or $-2+v$; the first substitution giving $x=-1+5v$ and $y=2-3v$; and the second $x=4+5v$ and $y=-1-3v$.

Exer. 5.

$7x-9y=5.$

$x=\frac{5+9y}{7}=y+\frac{5+2y}{7}.$

$\frac{5+2y}{7}=v_1$, and therefore $y=\frac{7v_1-5}{2}=3v_1-2+\frac{v_1-1}{2}.$

$\frac{v_1-1}{2}=v$, and therefore $v_1=2v+1.$

Hence,

$y=7v+1$, and $x=9v+2.$

v	-2,	-1,	0,	1,	2, ...
x	-16,	-7,	2,	11,	20, ...
y	-13,	-6,	1,	8,	15, ...

In employing the second method, we find $p=3$ and $q=4$; and, since $7x$ and $-9y$ have contrary signs, we have (§ 232.) $x=20+9v$ and $y=15+7v$. By changing v into $v-2$, we should get $x=9v+2$, and $y=7v+1$; which are the same as the values found by the first method.

Exer. 6.

$8x-7y=1.$

$y=\frac{8x-1}{7}=x+\frac{x-1}{7};$

$\frac{x-1}{7}=v$, and therefore $x=7v+1.$

and $y=8v+1.$

v	-2,	-1,	0,	1,	2, ...
x	-13,	-6,	1,	8,	15, ...
y	-15,	-7,	1,	9,	17, ...

In working by the second method, we find $p=1$, $y=1$, $x=1+7v$, and $y=1+8v$. These values are the same as those obtained above, and are the simplest that can be found.

Exer. 7.

By taking the second equation from three times the first, and dividing by 4, we get $y+2z=16$; whence $y=16-2z$: and, by

substituting this value of y in the first equation, transposing and contracting, we get $x=z-4$. Hence the answers are found by assuming z successively equal to 1, 2, 3, 4, &c., and finding the corresponding values of x and y.

Easy solutions may also be obtained by eliminating successively y and z. Thus, by eliminating y we should have $z-x=4$; while, by eliminating z, we should get $2x+y=8$: from each of which a solution may be obtained with great facility.

Exer. 8.

Here, to find positive answers, we may assume z successively equal to 1, 2, 3, ..., as long as we shall find that such assumptions will give positive values for x and y. Thus, by taking $z=1$, and by transposition, the given equation gives $2x+3y=17$; and, by either of the methods already employed, we find that the values of x are 7, 4, and 1; and that the corresponding values of y are 1, 3, and 5. Again, by taking $z=2$ in the given equation, we get $2x+3y=13$; and thence, in either of the usual modes, we find that the values of x are 5 and 2, and those of y, 1 and 3. In the third place, if we assume $z=3$, we get $2x+3y=9$; the only whole positive values of x and y belonging to which are 3 and 1. If, again, $z=4$, the given equation becomes $2x+3y=5$; an equation in which it will readily appear that the only whole positive values of x and y are 1 and 1. Here the solution ends, as by taking z equal to 5, 6, &c. it would be seen, that the resulting equations would give no simultaneous positive values of x and y.

Exer. 9.

Since the respective numbers of sixpences in 150*l.*, in 4*s.* 6*d.*, and in 21*s.* are 6000, 9, and 42, we have, by the question, and by dividing by 3, $3x+14y=2000$; where x denotes the number of dollars and y the number of guineas. From this we get

$$x=\frac{2000-14y}{3}=667-5y+\frac{y-1}{3}.$$

Then, by putting the fractional part of this latter expression $=v$, we get $y=3v+1$; and by substituting this in the foregoing value of x, we find $x=662-14v$. Hence, by taking $v=0$, we get 1 as the least positive value of y, and 662, the greatest one of x; and the succeeding values of y would be found by successive

additions of 3; while those of x would be obtained by continued subtractions of 14 from the first, 662. Now, by dividing 662 by 14, we get 47 as quotient with the remainder 4; whence it appears, that, besides 662 itself, there are 47 positive values for x, corresponding to positive values of y, and that the last of them is 4. The entire number of answers, therefore, is 48.

Exer. 10.

It is plain, that if 1 be added to any number which is a common multiple of 2, 3, 4, 5, and 6, the sum will answer all the conditions of this question except the last. Assuming x, therefore, to represent the required number, and, for simplicity, employing 60, the least common multiple of 2, 3, 4, 5, and 6, we have $x=60y+1$. By the last condition of the question, also, we have $x=7z$. Hence, therefore, $7z=60y+1$; from which, by dividing by 7, and otherwise following out the usual process, we readily find $y=7v-2$; and thence $7z$ or $x(=60y+1)=420v-119$. The answer in the form given in the ALGEBRA, is derived from this by changing v into $1+v$.

Exer. 11.

According to the method pointed out in the ALGEBRA, in the note in page 228., we have merely to determine the whole positive values of x and y in the equation $6x+11y=191$. Transposing $11y$, therefore, and dividing by 6, we get $x=32-2y+\frac{y-1}{6}$. Then, by putting the fractional part equal to v, we get $y=6v+1$, and, consequently, $x=30-11v$. Hence we have the first part $(=6x+5)=185-66v$, and the second $(=11y+4)=66v+15$; and the answers are found by taking v successively equal to 0, 1, and 2.

DIOPHANTINE ANALYSIS.

(ALGEBRA, p. 242.)

Exer. 1.

Here we have x^3+x^2 a square, and from this, by dividing by x^2, we get $x+1$, which must also be a square. Let it be assumed

$=v^2$: then $x=v^2-1$, where v may be of any value whatever. As examples, if $v=2$, we have $x=3$, and $x^3+x^2=36=6^2$; while if $v=3$, we have $x=8$, and $x^3+x^2=512+64=576=24^2$. It is easy to see that in general terms, $x^3+x^2=(x+1)x^2=v^2(v^2-1)^2$.

Exer. 2.

By making the assumption pointed out in the note in the ALGEBRA, page 243., we satisfy the first and second conditions of the question, x^2+2x+1 and x^2-2x+1 being each a square. Then, the sum and difference of the assumed quantities being $2x^2$ and $4x$, it remains, that we find such values of x as shall make $2x^2+1$ and $4x+1$ squares. Assume the latter $=v^2$; then $x=\frac{v^2-1}{4}$. Let this be substituted for x in the former expression $2x^2+1$, and the product of the result by the square number 16, is $2v^4-4v^2+18$, which must be a square; and, by trial, we find that it is such, when $v=1$. Substituting, therefore, $y+1$ for v, according to § 243., we get $2y^4+8y^3+8y^2+16$. To make this a square, assume it $=(v_1y^2+4)^2$* $=v_1^2y^4+8v_1y^2+16$. Then, by rejecting 16, and equalling $8v_1$ and 8, the coefficients of y^2, we get $v_1=1$, and consequently $2y^4+8y^3+8y^2=y^4+8y^2$; whence, by rejecting $8y^2$, &c., we get $y=-8$; and thence $v(=y+1)=-7$. Hence $x\left(=\frac{v^2-1}{4}\right)=12$, and consequently $x^2+2x=168$, and $x^2-2x=120$.

Exer. 3.

Let x and $x+1$ represent the required numbers; then, by taking the difference of the cubes of these we get $3x^2+3x+1$, which by the question is to be a square. Let this be assumed equal to $(vx-1)^2$, and it will be found by easy operations, that $x=\frac{2v+3}{v^2-3}$, and consequently $x+1=\frac{v^2+2v}{v^2-3}$. The solution would be rendered rather more general by assuming x and $x+y^2$ to represent the required numbers. In proceeding in this way

* We are plainly at liberty to omit in this the term v_2y, as there is no term containing y in $2y^4+8y^3+8y^2+16$.

the difference of the cubes would be divisable by y^2, and the rest of the work would be easy. The values of the required numbers would thus be found to be the same as the products of those above obtained, when multiplied by y^2; results which we should naturally anticipate without going through with a new investigation.

Exer. 4.

The assumption proposed in the note in the ALGEBRA, page 243., satisfies one of the conditions, since the difference of the squares of the assumed quantities is $8a^6x^3$, which is a cube (the cube of $2a^2x$). Again, the difference of the cubes of the same quantities is $12a^6x^6+16a^{18}$, which is to be a square; and therefore, if this be divided by $4a^6$, which is a square, the quotient $3x^6+4a^{12}$ will be a square. Assuming this equal to $(vx^3+2a^6)^2$, we readily find $x^3=\dfrac{4va^6}{3-v^2}$. Now this becomes $8a^6$, when v is taken $=-2$, or $\dfrac{3}{2}$; and being a cube (the cube of $2a^2$), it answers. We find therefore, the required numbers, by adding and subtracting $2a^6$, to be $10a^6$ and $6a^6$, where a may be taken of any value whatever.

Exer. 5.

By the substitution pointed out in the ALGEBRA, in one of the notes in page 243., we get $2y^2+4y$, which is to be a square. Let it be assumed $=v^2y^2$, and it will be readily found that $y=\dfrac{4}{v^2-2}$; and consequently, $x=y+1=\dfrac{v^2+2}{v^2-2}$.

The same answer may also be found by assuming $2x^2-2$, or its equal $(2x+2)(x-1)=v^2(x-1)^2$; and the same still, only with the opposite sign*, by assuming $2x^2-2$, or, what is the same, $(2x-2)(x+1)=v^2(x+1)^2$.

Exer. 6.

It will be found by trial, that the proposed expression is a square, when $x=3$. Substituting, therefore, $y+3$ for x, we get $13y^2+93y+169$; and by assuming this $=(vy+13)^2$, we readily

* This makes no difference in the value of $2x^2-2$, $2x^2$ being the same whether x is positive or negative.

find $y=\frac{93-26v}{v^2-13}$, and consequently $x=y+3=\frac{3v^2-26v+54}{v^2-13}$. In this, to get positive values of x, we may take v equal to 6, 7, 8, &c., or to -4, -5, -6, &c.

Exer. 7.

By the assumption proposed in the note in the ALGEBRA on this question, one condition of the problem is satisfied, since $(x-1)^2+4x=x^2+2x+1=(x+1)^2$. It only remains, therefore, to make $(4x)^2+x-1$, or $16x^2+x-1$, a square; and this will be effected by assuming it $=(4x-v)^2=16x^2-8vx+v^2$, which gives $x=\frac{v^2+1}{8v+1}$. From this the expressions given in the ALGEBRA for the required numbers ($4x$ and $x-1$) will be easily obtained.

Solutions for this question may be obtained by means of numberless other assumptions. Thus, we might represent the required numbers by x and $2x+1$, by x and $-2x+1$ by x and $\pm 4x+4$, by $2x$ and $\pm 4x+1$, &c., &c.

Exer. 9.

By assuming, with Bonnycastle, $4x$, x^2-4x, and $2x+1$, to represent the required numbers, we have the sum of all three $=x^2+2x+1=(x+1)^2$, the sum of the first and second $=x^2$, and the sum of the second and third $=x^2-2x+1=(x-1)$, which are all squares. It only remains, therefore, that we make $6x+1$, the sum of the first and third, a square. To do this, assume it $=v^2$: then $x=\frac{1}{6}(v^2-1)$; and the answers are found by substituting this for x in the three quantities originally assumed.

Answers of a different form would be found by assuming $6x+1=(vx\pm 1)^2$.

Exer. 10.

By putting the proposed quantity $=(vx-2)^3$, and rejecting the last terms, we get

$$6x^3+9x^2+36x=v^3x^3-6v^2x^2+12vx.$$

Hence, by putting $12v=36$, and consequently $v=3$, we can reject the last terms; and, by dividing by x^2, &c., we get $x=3$.

Exer. 11.

By assuming the proposed quantity equal to $(vx^2+v'x+4)^2$, and by actual squaring, and rejecting the last terms, we get

$$3x^4-10x^3+24x^2+32x=v^2x^4+2vv'x^3+(v'^2+8v)x^2+8v'x.$$

Then, by taking $8v'=32$, and $v'^2+8v=24$, we find $v'=4$, and $v=1$; and by rejecting the last two terms of each member, dividing by x^3, and taking v and v' as found above, we get $3x-10=x+8$; whence $x=9$.

Exer. 12.

Let x and y be assumed to denote the numbers: then $x^3+y^3=x+y$; or, by dividing by $x+y$, $x^2-xy+y^2=1$. By resolving this as a quadratic, we get $x=\frac{1}{2}\{y\pm\sqrt{(4-3y^2)}\}$, an expression in which y may be taken of any value whatever, and which, whether real or imaginary, will always answer.

When this question, however, is regarded as belonging to the Diophantine Analysis, y must be such as to render the term $\sqrt{(4-3y^2)}$ rational. To effect this, assume $\sqrt{(4-3y^2)}=2-vy$. Then, by squaring, &c., we find $y=\dfrac{4v}{v^2+3}$; and, multiplying this by v, and taking the product from 2, we get $\sqrt{(4-3y^2)}=\dfrac{6-2v^2}{v^2+3}$. Lastly, by substituting this, and the value just found for y, in the expression, $x=\frac{1}{2}\{y\pm\sqrt{(4-3y^2)}\}$, we get $x=\dfrac{2v\pm(3-v^2)}{v^2+3}=\dfrac{2v\mp(v^2-3)}{v^2+3}$.

As particular examples, let $v=1$; then $x=1$ or 0, and $y=1$: let $v=2$; then $x=\frac{3}{7}$ or $\frac{5}{7}$, and $y=\frac{8}{7}$, &c.

Exer. 13.

Here, by assuming $2(x^2+x)$ to denote the required number, we satisfy one condition of the question, since, if one be added to the double of this, we get $4x^2+4x+1$, which is the square of $2x+1$. Then, by multiplying the assumed value by 3 and adding unity to the product, we get $6x^2+6x+1$, which is to be a square. To find the value of x which will make it such, as-

sume it $=(vx-1)^2$; and it will be readily found, that $x=\frac{2v+6}{v^2-6}$. Hence, we have $x+1=\frac{v^2+2v}{v^2-6}=\frac{v(v+2)}{v^2-6}$. We have also $2x=\frac{2(2v+6)}{v^2-6}=\frac{4(v+3)}{v^2-6}$. Then, by taking the product of these, we have the required number, $2(x^2+x)$ or $2x(x+1)=\frac{4v(v+2)(v+3)}{(v^2-6)^2}$, a form for the answer which is perhaps rather preferable to that which is given in the ALGEBRA.

In the following table, the second line exhibits particular answers; and the first line, the values of v which produce them:

1,	2,	3,	4,	5,	6,	$2\frac{1}{2}$,	$2\frac{9}{20}$,	&c.
$\frac{48}{25}$,	40,	40,	6·72,	$\frac{1120}{361}$,	1·92,	3960,	38027920,	&c.

Exer. 14.

The solution of this question is found simply by taking $a=2$ and $b=9$, in the value obtained for x in Exam. 3., in the ALGEBRA, page 240. In this way we find $x=\frac{9(v^2-1)+4v}{v^2+1}$: and, by assigning to v the values in the first line of the following table, we get, for the corresponding values of x^2 and $85-x^2$, the numbers in the second and third lines:

v	1,	2,	3,	7,	&c.
x^2	2^2,	7^2,	$8{\cdot}4^2$,	$9{\cdot}2^2$,	&c.
$85-x^2$	9^2,	6^2,	$3{\cdot}8^2$,	$0{\cdot}6^2$,	&c.

Exer. 15.

Here, if we denote the required numbers by x and y, we have $x^2-y^2=x-y$; and hence, if we divide by $x-y$, we get $x+y=1$, so that one condition necessary for the solution of the problem is that the sum of the numbers shall be unity. The numbers, therefore, are x and $1-x$, and the sum of the squares of these is $2x^2-2x+1$, which, by the question, is to be a square. Let it be assumed $=(vx-1)^2$, and it will be readily found that

$x=\frac{2v-2}{v^2-2}$; and consequently $1-x=1-\frac{2v-2}{v^2-2}=\frac{v^2-2v}{v^2-2}$.

The following table exhibits particular answers: —

v	2,	3,	4,	5,	&c.
x	1,	$\frac{4}{7}$,	$\frac{4}{7}$,	$\frac{8}{23}$,	&c.
$1-x$	0,	$\frac{3}{7}$,	$\frac{3}{7}$,	$\frac{15}{23}$,	&c.

SERIES.

(Algebra, p. 249.)

Exer. 1.

In this question the general term is plainly $\frac{1}{(n+3)(n+4)}$, or what is equivalent, $\frac{1}{n+3}-\frac{1}{n+4}$. Hence, by taking n successively equal to 1, 2, 3, &c., and by placing the positive terms in the first line and the negative in the second, we get

$$S_n=\left\{\begin{array}{l}\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\ldots+\frac{1}{n+3}\\ \quad-\frac{1}{5}-\frac{1}{6}-\frac{1}{7}-\ldots-\frac{1}{n+3}-\frac{1}{n+4}\end{array}\right\};$$

$$\text{or } S_n=\frac{1}{4}-\frac{1}{n+4}=\frac{n}{4(n+4)}.$$

When $n=\infty$, we have from the first form of the sum, $S_\infty=\frac{1}{4}$, the sum of an infinite number of terms of the proposed series.

Exer. 2.

In this exercise the general term is evidently $\frac{1}{(n+1)(n+2)(n+3)}$, or, by § 255., $\frac{1}{2}\left\{\frac{1}{(n+1)(n+2)}-\frac{1}{(n+2)(n+3)}\right\}$. By taking in this n successively equal to 1, 2, 3, &c., and placing the positive and negative terms in different lines, we get

$$S_n=\frac{1}{2}\left\{\begin{array}{l}\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\frac{1}{4\cdot5}+\ldots+\frac{1}{(n+1)(n+2)} \\ \quad -\frac{1}{3\cdot4}-\frac{1}{4\cdot5}-\ldots-\frac{1}{(n+1)(n+2)}-\frac{1}{(n+2)(n+3)}\end{array}\right\}$$

$$\text{or } S_n=\frac{1}{2}\left\{\frac{1}{2\cdot3}-\frac{1}{(n+2)(n+3)}\right\}=\frac{1}{12}-\frac{1}{2}\,\frac{1}{(n+2)(n+3)}.$$

Hence, by taking $n=\infty$, we get $S_\infty=\frac{1}{12}$.

Exer. 3.

Here the general term is

$$\frac{1}{n(n+1)(n+2)(n+3)}, \text{ or, § 255, } \frac{1}{3}\left\{\frac{1}{n(n+1)(n+2)}-\frac{1}{(n+1)(n+2)(n+3)}\right\}$$

Then, by taking n in the latter form, successively equal to 1, 2, 3, &c., and arranging the results in the usual manner, we get

$$S_n=\frac{1}{3}\left\{\begin{array}{l}\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\ldots+\frac{1}{n(n+1)(n+2)} \\ \quad -\frac{1}{2\cdot3\cdot4}-\frac{1}{3\cdot4\cdot5}-\ldots-\frac{1}{n(n+1)(n+2)}-\frac{1}{(n+1)(n+2)(n+3)}\end{array}\right\}$$

$$\text{or } S_n=\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)};$$

whence $S_\infty=\frac{1}{18}$.

Exer. 4.

In this exercise, the general term is plainly $\frac{1}{n(n+2)}$, or by § 254., $\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right)$. Then, by taking n successively equal to 1, 2, 3, &c., this becomes $\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right)$, $\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right)$, $\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)$, &c.; and, by the usual arrangement, we have the following results;

$$S_n = \frac{1}{2}\left\{\begin{array}{l}\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots+\frac{1}{n} \\ \qquad -\frac{1}{3}-\frac{1}{4}-\ldots\ldots-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}\end{array}\right\};$$

$$\text{or } S_n = \frac{3}{4}-\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}\right)=\frac{3}{4}-\frac{2n+3}{2(n^2+3n+2)};$$

whence $S_\infty = \dfrac{3}{4}$.

Exer. 5.

Here the general term is

$$\frac{1}{n(n+3)}, \text{ or, by § 254., } \frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+3}\right).$$

Accordingly, the first, second, third, &c., terms are $\frac{1}{3}\left(\frac{1}{1}-\frac{1}{4}\right)$, $\frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)$, $\frac{1}{3}\left(\frac{1}{3}-\frac{1}{6}\right)$, $\frac{1}{3}\left(\frac{1}{4}-\frac{1}{7}\right)$, &c.; and, by following the usual arrangement, we shall have

$$S_n = \frac{1}{3}\left\{\begin{array}{l}\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots+\frac{1}{n} \\ \qquad -\frac{1}{4}-\frac{1}{5}-\ldots\ldots-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\end{array}\right\};$$

or, by contraction,

$$S_n = \frac{11}{18}-\frac{1}{3}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right);$$

whence $S_\infty = \dfrac{11}{18}$.

Exer. 6.

This question may be solved very simply in the following manner: First, suppose the number of terms to be even; then the series will be

$$\frac{2}{3\cdot5}-\frac{3}{5\cdot7}+\frac{4}{7\cdot9}-\frac{5}{9\cdot11}+\ldots+\frac{n}{(2n-1)(2n+1)}-\frac{n+1}{(2n+1)(2n+3)};$$

or, by taking the second term from the first, the fourth from the third, &c.,

$$\frac{1}{3\cdot 7}+\frac{1}{7\cdot 11}+\frac{1}{11\cdot 15}+\ldots\ldots+\frac{1}{(2n-1)(2n+3)}.$$

Hence, by expressing each fraction as the difference of two others, according to § 254., and by the usual arrangement, we have

$$S_n=\frac{1}{4}\left\{\begin{array}{l}\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\ldots+\frac{1}{2n-5}+\frac{1}{2n-1}\\ -\frac{1}{7}-\frac{1}{11}-\ldots-\frac{1}{2n-5}-\frac{1}{2n-1}-\frac{1}{2n+3}\end{array}\right\}=\frac{1}{12}-\frac{1}{4(2n+3)}.$$

To find the sum of an odd number of terms, let n be changed into $n-1$ in the answer now found, and to the result add the n^{th} term $\frac{n+1}{(2n+1)(2n+3)}$: then

$$S_n=\frac{1}{12}-\frac{1}{4(2n+3)}+\frac{n+1}{(2n+1)(2n+3)},\text{ or } S_n=\frac{1}{12}+\frac{1}{4(2n+3)},$$

by the actual incorporation of the last two terms. The two answers are combined in one by employing the factor $(-1)^n$, as is done in the ALGEBRA, this factor being positive, when n is even, and negative, when it is odd. The sum of an infinite number of terms is evidently $\frac{1}{12}$.

(ALGEBRA, p. 254.)

Exer. 7.

In dividing 1 by $1+2x+3x^2+$ &c., we get $q_1=1$, with the remainder $-2x-3x^2-4x^3-$ &c. Dividing $1+2x+3x^2+$ &c. by this remainder, we find $q_2=-\frac{1}{2x}$, with the remainder $\frac{x}{2}+\frac{2x^2}{2}+\frac{3x^3}{2}+$ &c. Taking this remainder as divisor, and $-2x-3x^2-4x^3-$ &c. as dividend, we get $q_3=-4$, and the remainder is $x^2+2x^3+3x^4+$ &c. Lastly, dividing $\frac{x}{2}+\frac{2x^2}{2}+\frac{3x^3}{2}+$

&c. by this remainder, we get $\frac{1}{2x}$ as q_4, with no remainder. Hence, the continued fraction is

$$\frac{1}{1+\dfrac{1}{\left(-\dfrac{1}{2x}\right)+\dfrac{1}{-4+\dfrac{1}{\left(\dfrac{1}{2x}\right)}}}}\ ;\ \text{or}\ \frac{1}{1}-\frac{2x}{1}+\frac{2x}{4}-\frac{2}{1}.$$

The following is the method of finding (§ 216.), the value of the continued fraction in its first form:

$$1,\quad -\frac{1}{2x},\qquad -4,\qquad \frac{1}{2x},$$

$$\frac{1}{1},\ \frac{\left(-\dfrac{1}{2x}\right)}{1-\dfrac{1}{2x}},\ \frac{1+\dfrac{2}{x}}{-3+\dfrac{2}{x}},\ \frac{-\dfrac{1}{2x}+\dfrac{1}{2x}+\dfrac{1}{x^2}}{1-\dfrac{1}{2x}-\dfrac{3}{2x}+\dfrac{1}{x^2}}=\frac{1}{x^2-2x+1}=\frac{1}{(1-x)^2}.$$

For finding the required value from the second form of the continued fraction, we have the value expressed by the last two of the component fractions $=\frac{2x}{4-2x}=\frac{x}{2-x}$; and if this be attached to the denominator of the preceding fraction, there results

$$-\frac{2x}{1+\dfrac{x}{2-x}},\ \text{or}\ -2x+x^2,$$

by multiplying the numerator and denominator by $2-x$, and contracting. Lastly, by attaching this to the denominator of the remaining fraction, we get $\frac{1}{1-2x+x^2}$, as before.

Exer. 8.

Here, by dividing 1 by $x+x^2-x^4-x^5+x^7+$ &c., we get $q_1=\frac{1}{x}$, with the remainder, $-x+x^3+x^4-x^6-x^7+x^9+$ &c.; and in dividing $x+x^2-x^4-$ &c. by this, we get $q_2=-1$, with the remainder $x^2+x^3-x^5-x^6+x^8+x^9-$ &c. By a like division,

q_3 is found to be $-\frac{1}{x}$, and the corresponding remainder is $x^2+x^3-x^5-x^6+x^8+x^9-$ &c., the same as the last divisor, so that we get $q_4=1$, with no remainder. Hence the continued fraction is

$$\cfrac{1}{\left(\frac{1}{x}\right)+\cfrac{1}{-1+\cfrac{1}{\left(\frac{-1}{x}\right)+\frac{1}{1}}}}, \text{ or } \frac{x}{1}-\frac{x}{1}+\frac{x}{1}-\frac{x}{1}.$$

The work for finding the value of the continued fraction in its first form is as follows: —

$$\frac{1}{x},\quad -\frac{1}{1},\quad -\frac{1}{x},\quad \frac{1}{1},$$

$$\frac{1}{\left(\frac{1}{x}\right)},\quad \frac{\left(-\frac{1}{1}\right)}{1-\frac{1}{x}},\quad \frac{1+\frac{1}{x}}{\left(\frac{1}{x^2}\right)},\quad \frac{-\frac{1}{1}+1+\frac{1}{x}}{1-\frac{1}{x}+\frac{1}{x^2}}=\frac{x}{x^2-x+1}.$$

In finding the value from the second form of the continued fraction, we have the last two component fractions equivalent to $\frac{x}{1-x}$. Adding this to the denominator of the preceding fraction, and multiplying the numerator and denominator of the result by $1-x$, we get $-\frac{x-x^2}{1}$ or $-x+x^2$; by attaching which to 1, the denominator of the first component fraction, we get $\frac{x}{1-x+x^2}$, the same as befor .

Exer. 9.

By dividing the cosine by the sine, we get $q_1=\frac{1}{x}$, with the remainder $-\frac{x^3}{3}+\frac{x^4}{30}-\frac{x^5}{840}+$ &c. By dividing the sine by this, we find $q_2=-\frac{3}{x}$, with the remainder $-\frac{x^3}{15}+\frac{x^5}{210}-$ &c.; and,

by employing more terms of the series, and continuing the process in the usual way, we readily get $q_3=\frac{5}{x}$, $q_4=-\frac{7}{x}$, $q_5=\frac{9}{x}$, &c. Hence the continued fraction will be

$$\cfrac{1}{\left(\frac{1}{x}\right)+\cfrac{1}{\left(-\frac{3}{x}\right)+\cfrac{1}{\left(\frac{5}{x}\right)+\cfrac{1}{\left(-\frac{7}{x}\right)+\cfrac{1}{\left(\frac{9}{x}\right)+\text{ \&c.;}}}}}}$$

$$\text{or}\quad \cfrac{x}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\cfrac{x^2}{7-\cfrac{x^2}{9-\text{ \&c.}}}}}}$$

The work for finding the converging fractions from the first of these forms is as follows: —

$$\begin{array}{c|c|c|c|c|c}
\frac{1}{x} & -\frac{3}{x} & \frac{5}{x} & -\frac{7}{x} & \frac{9}{x} & \text{\&c.} \\[2ex]
\dfrac{1}{\left(\frac{1}{x}\right)} & \dfrac{\left(-\frac{3}{x}\right)}{1-\frac{3}{x^2}} & \dfrac{1-\frac{15}{x^2}}{\frac{1}{x}+\frac{5}{x}-\frac{15}{x^3}} & \dfrac{-\frac{3}{x}-\frac{7}{x}+\frac{105}{x^3}}{1-\frac{3}{x^2}-\frac{42}{x^2}+\frac{105}{x^4}} & \dfrac{1-\frac{15}{x^2}-\frac{90}{x^2}+\frac{945}{x^4}}{\frac{6}{x}-\frac{15}{x^3}+\frac{9}{x}-\frac{405}{x^3}+\frac{945}{x^5}} & \text{\&c.} \\[2ex]
 & =\dfrac{3x}{3-x^2} & =\dfrac{1-\frac{15}{x^2}}{\frac{6}{x}-\frac{15}{x^3}} & =\dfrac{\frac{-10}{x}+\frac{105}{x^3}}{1-\frac{45}{x^2}+\frac{105}{x^4}} & =\dfrac{1-\frac{105}{x^2}+\frac{945}{x^4}}{\frac{15}{x}-\frac{420}{x^3}+\frac{945}{x^5}} & \text{\&c.} \\[2ex]
 & & =\dfrac{x^3-15x}{6x^2-15} & =\dfrac{105x-10x^3}{105-45x^2+x^4} & =\dfrac{945x-105x^3+x^5}{945-420x^2+15x^4} & \text{\&c.}
\end{array}$$

The same results are very easily found from the second form of the continued fraction, by commencing with any term, suppose $-\frac{x^2}{9}$, and going backward, step by step, till the value of the

continued fraction up to that point is expressed in a common fraction.*

Exer. 10.

By § 212., we have

$$(1+x)^n=1+nx+\frac{n(n-1)}{1\cdot2}x^2+\frac{n(n-1)(n-2)}{1\cdot2\cdot3}x^3+\text{ \&c.}$$

Dividing 1 by this, we get $q_1=1$, with the remainder $-nx-\frac{n(n-1)x^2}{1\cdot2}-$ &c.; and if by this we divide $1+nx+\frac{n(n-1)x^2}{1\cdot2}+$ &c., we find $-\frac{1}{nx}$ as the value of q_2. Continuing the division in the usual way, we obtain

$$q_3=-\frac{2n}{n+1},\quad q_4=\frac{3(n+1)}{n(n-1)x},\quad q_5=\frac{2n(n-1)}{(n+1)(n+2)},\text{ \&c.}$$

Hence the continued fraction is

$$\frac{1}{1}+\cfrac{1}{\left(-\frac{1}{nx}\right)+\cfrac{1}{\left(-\frac{2n}{n+1}\right)+\cfrac{1}{\left\{\frac{3(n+1)}{n(n-1)x}\right\}+\cfrac{1}{\left\{\frac{2n(n-1)}{(n+1)(n+2)}\right\}+\text{\&c.};}}}}$$

$$\text{or }\cfrac{1}{1-\cfrac{nx}{1+\cfrac{\frac{1}{2}(n+1)x}{1-\cfrac{\frac{1}{6}(n-1)x}{1+\cfrac{\frac{1}{6}(n+2)x}{1-\text{\&c.}}}}}}$$

---

* Thus,

$$\cfrac{x}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\cfrac{x^2}{7-\cfrac{x^2}{9}}}}}=\cfrac{x}{1-\cfrac{x^2}{3-\cfrac{x^2}{5-\cfrac{9x^2}{63-x^2}}}}=\cfrac{x}{1-\cfrac{x^2}{3-\cfrac{63x^2-x^4}{315-14x^2}}}$$

$$=\cfrac{x}{1-\cfrac{315x^2-14x^4}{945-105x^2+x^4}}=\frac{945x-105x^3+x^5}{945-420x^2+15x^4}.$$

The converging fractions obtained in either of these ways, will give the tangent of any assigned arc, with a much greater approach to accuracy than would be obtained by using only the same powers of x in the series, from which the continued fraction is derived.

Exer. 11.

By employing the first five of the quotients in Exam. 6., the work, conducted in the usual way, will stand as follows: —

$$1;\quad -\frac{1}{x};\quad -2;\quad \frac{3}{x};$$

$$\frac{1}{1};\quad \frac{-\frac{1}{x}}{1-\frac{1}{x}};\quad \frac{1+\frac{2}{x}}{1-2+\frac{2}{x}},\ \text{or}\ \frac{1+\frac{2}{x}}{-1+\frac{2}{x}};\quad \frac{-\frac{1}{x}+\frac{3}{x}+\frac{6}{x^2}}{1-\frac{1}{x}-\frac{3}{x}+\frac{6}{x^2}},\ \text{or}\ \frac{\frac{2}{x}+\frac{6}{x^2}}{1-\frac{4}{x}+\frac{6}{x^2}};$$

$$2;$$

$$\frac{1+\frac{2}{x}+\frac{4}{x}+\frac{12}{x^2}}{-1+\frac{2}{x}+2-\frac{8}{x}+\frac{12}{x^2}},\ \text{or}\ \frac{1+\frac{6}{x}+\frac{12}{x^2}}{1-\frac{6}{x}+\frac{12}{x^2}}.$$

The last of these expressions, by inverting the order of its terms, and multiplying the numerator and denominator by x^2, gives the answer in its proper form.*

Exer. 12.

In finding the converging fractions by means of the first six quotients in Exam. 7., the work will be as follows: —

* We may very readily obtain the same result by employing the first five of the component fractions in the second form of the continued fraction in Exam. 6., commencing with the last, and using it and the others in the reversed order. Thus, by multiplying the numerator and denominator by 2, we get, for the fourth and fifth $\frac{2}{6+x}$. If this be attached to the denominator of the third, and the numerator and denominator of the result be multiplied by $6+x$, there is obtained $\frac{6x+x^2}{12}$. Attaching this to the denominator of the second fraction, and multiplying in a similar manner by 12, we get $\frac{12x}{12+6x+x^2}$. Lastly, by connecting this with the denominator of the first fraction, and by a like process, we get the answer.

$$\frac{1}{x},\quad \frac{2}{1},\quad \frac{3}{x},\quad \frac{2}{2},\quad \frac{5}{x},$$

$$\frac{1}{\left(\frac{1}{x}\right)},\quad \frac{2}{1+\frac{2}{x}},\quad \frac{1+\frac{6}{x}}{\frac{1}{x}+\frac{7}{x}+\frac{6}{x^2}},\quad \frac{2+1+\frac{6}{x}}{1+\frac{2}{x}+\frac{4}{x}+\frac{6}{x^2}},\quad \frac{1+\frac{6}{x}+\frac{15}{x}+\frac{30}{x^2}}{\frac{4}{x}+\frac{6}{x^2}+\frac{5}{x}+\frac{30}{x^2}+\frac{30}{x^3}},$$

$$\frac{2}{3},$$

$$\frac{3+\frac{6}{x}+\frac{2}{3}+\frac{14}{x}+\frac{20}{x^2}}{1+\frac{6}{x}+\frac{6}{x^2}+\frac{6}{x}+\frac{24}{x^2}+\frac{20}{x^3}}.$$

From this last result, the answer will be obtained by contracting, and by multiplying the numerator and denominator by $3x^3$.

Exer. 13.

By dividing 1 by the given series, that series by the remainder, and so on in the usual way, we find the successive quotients to be x, $\frac{1}{3}$, $-\frac{9}{4}x$, and $-\frac{4}{3}$; with the last of which there is no remainder. The rest of the work will stand as follows:—

$$x,\quad \frac{1}{3},\quad -\frac{9}{4}x\quad -\frac{4}{3},$$

$$\frac{1}{x},\quad \frac{\frac{1}{3}}{x+\frac{1}{3}},\quad \frac{1-\frac{3}{4}x}{x-\frac{9}{4}x-\frac{3}{4}x^2},\quad \frac{\frac{1}{3}-\frac{4}{3}+x}{1-\frac{1}{3}x+\frac{5}{3}x+x^2}.$$

The last of these fractions becomes, by contraction, $\frac{x-1}{x^2+2x+1}$, the answer.

Exer. 14.

As the solution of this question by the method of continued fractions is very laborious, on account of the large numbers which arise in the course of the operation, the following solution is given in preference:

$$\text{Let } s = x + 8x^2 + 27x^3 + 64x^4 + 125x^5 + \text{ \&c.},$$

and multiply both members by $1-x$: then

$$s(1-x) = x + 7x^2 + 19x^3 + 37x^4 + 61x^5 + \text{ \&c.},$$

Multiply, again, by $1-x$: then

$$s(1-x)^2 = x + 6x^2 + 12x^3 + 18x^4 + 24x^5 + \text{ \&c.}.$$

By two similar successive multiplications we get

$$s(1-x)^3 = x + 5x^2 + 6x^3 + 6x^4 + 6x^5 + \text{ \&c.}; \text{ and}$$
$$s(1-x)^4 = x + 4x^2 + x^3;$$

and s, the sum of the series, is found from the latter equation by dividing by $(1-x)^4$.

Exer. 15.

Here assume

$$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \mathrm{A}n + \mathrm{B}n^2 + \mathrm{C}n^3 + \mathrm{D}n^4.$$

Then, by changing n into $n+1$, and taking the foregoing equation from the result, we get

$$(n+1)^3 = \mathrm{A} + \mathrm{B}(2n+1) + \mathrm{C}(3n^2+3n+1)$$
$$+ \mathrm{D}(4n^3 + 6n^2 + 4n + 1).$$

Hence, by expanding $(n+1)^3$, &c., and by equalling the coefficients of the like powers of n, we get

$$4\mathrm{D} = 1,\ 6\mathrm{D} + 3\mathrm{C} = 3,\ 4\mathrm{D} + 3\mathrm{C} + 2\mathrm{B} = 3, \text{ and}$$
$$\mathrm{D} + \mathrm{C} + \mathrm{B} + \mathrm{A} = 1.$$

The first of these equations gives $\mathrm{D} = \frac{1}{4}$, the second $\mathrm{C} = \frac{1}{2}$, the third $\mathrm{B} = \frac{1}{4}$, and the fourth $\mathrm{A} = 0$. Hence the required sum is

$$\tfrac{1}{4}n^2 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^4 = \tfrac{1}{4}(n^2 + 2n^3 + n^4) =$$
$$\{\tfrac{1}{2}(n+n^2)\}^2 = \{\tfrac{1}{2}n(1+n)\}^2.$$

Exer. 16.

Here, by assuming

$$1^4+2^4+3^4+\ldots\ldots+n^4=An+Bn^2+Cn^3+Dn^4+En^5,$$

by changing n into $n+1$, and taking the members of the first result from those of the second, we get

$$n^4+4n^3+6n^2+4n+1=A+B(2n+1)+C(3n^2+3n+1)+D$$
$$(4n^3+6n^2+4n+1)+E(5n^4+10n^3+10n^2+5n+1).$$

Hence, by equalling the coefficients of the like powers of n, we get

$$5E=1,\ 10E+4D=4,\ 10E+6D+3C=6,$$
$$5E+4D+3C+2B=4, \text{ and } E+D+C+B+A=1:$$

and from these in succession we find $E=\frac{1}{5}$, $D=\frac{1}{2}$, $C=\frac{1}{3}$, $B=0$, and $A=-\frac{1}{30}$. Using these values, therefore, for the assumed coefficients, we get the required sum.

Exer. 17.

Putting $1^5+2^5+\ldots\ldots+n^5=An+Bn^2+Cn^3+Dn^4+En^5+Fn^6$; then changing n into $n+1$, subtracting the members of the assumed equation from those of the result, and equalling the coefficients of the like powers of n, we get the answer in the usual way, without difficulty.

Exer. 18.

To solve this question let us assume

$$1^2+3^2+5^2+\ldots\ldots+(2n-1)^2=An+Bn^2+Cn^3.$$

Then, by changing n into $n+1$, and by taking the members of the foregoing equation from those of the result, we get

$$(2n+1)^2=A+B(2n+1)+C(3n^2+3n+1).$$

From this, by actually performing the operations indicated, and equalling the coefficients of the like powers of n, we get

$$3C=4,\ 3C+2B=4, \text{ and } C+B+A=1\ ;$$

whence we find $C=\frac{4}{3}$, $B=0$, and $A=-\frac{1}{3}$. The sum, therefore, is $\frac{4}{3}n^3-\frac{1}{3}n$, or $\frac{1}{3}n(4n^2-1)$.

Exer. 19.

By assuming the given series equal to $An+Bn^2+Cn^3$, by changing n into $n+1$, and, by the usual subtraction, we get

$$(n+1)(n+2)=A+B(2n+1)+C(3n^2+3n+1).$$

Then, by performing the operations which are indicated, and equalling the corresponding coefficients, we obtain

$$3C=1, \qquad 3C+2B=3, \qquad \text{and } C+B+A=2;$$

the resolution of which equations gives $C=\frac{1}{3}$, $B=1$, and $A=\frac{2}{3}$. The answer, therefore, is $\frac{2}{3}n+n^2+\frac{1}{3}n^3$, or $\frac{1}{3}(2n+3n^2+n^3)$.

Exer. 20.

Assuming here

$$1\cdot2^2+2\cdot3^2+3\cdot4^2+\ldots\ldots+n(n+1)^2=An+Bn^2+Cn^3+Dn^4;$$

then changing n into $n+1$, and subtracting, as usual, we obtain

$$(n+1)(n+2)^2=A+B(2n+1)+C(3n^2+3n+1)$$
$$+D(4n^3+6n^2+4n+1).$$

Hence, by actually performing the operations, and equalling the coefficients of the like powers of n, we get

$$4D=1,\ 6D+3C=5,\ 4D+3C+2B=8,$$
$$\text{and } D+C+B+A=4:$$

and therefore $D=\frac{1}{4}$, $C=\frac{7}{6}$, $B=\frac{7}{4}$, and $A=\frac{5}{6}$. Hence the required sum is $\frac{5}{6}n+\frac{7}{4}n^2+\frac{7}{6}n^3+\frac{1}{4}n^4$.

Exer. 21.

To solve this question, assume

$$1\cdot3^2+3\cdot5^2+5\cdot7^2+\ \ldots\ldots\ +(2n-1)(2n+1)^2=An+Bn^2+$$
$$Cn^3+Dn^4.$$

Then, by changing n into $n+1$, and by the usual subtraction, we get

$$(2n+1)(2n+3)^2=A+B(2n+1)+C(3n^2+3n+1)$$
$$+D(4n^3+6n^2+4n+1).$$

From this, by performing the actual multiplications, and by equalling the corresponding coefficients, we obtain the following equations:

$$4D=8,\ 6D+3C=28,\ 4D+8C+2B=30,$$
$$\text{and } D+C+B+A=9.$$

From these, by easy operations, we find

$$D=2,\ C=\tfrac{16}{3},\ B=3, \text{ and } A=-\tfrac{4}{3}.$$

The answer, therefore, is

$$2n^4+\tfrac{16}{3}n^3+3n^2-\tfrac{4}{3}n, \text{ or}$$
$$2n^4+3n^2+\tfrac{1}{3}(16n^3-4n),$$
$$\text{or finally, } n^2(2n^2+3)+\tfrac{1}{3}n(16n^2-4).$$

APPLICATION OF ALGEBRA IN INVESTIGATIONS IN GEOMETRY.*

(Algebra, p. 265.)

Exer. 1.

If x be assumed to denote the side of a square, its perimeter will be $4x$, and (Euc. I. 47.) its diagonal $\sqrt{2x^2}$ or (Algebra, § 101.) $x\sqrt{2}$. Hence, by the question, $4x+x\sqrt{2}=s$. Hence, by dividing by $4+\sqrt{2}$, we get $x=\frac{s}{4+\sqrt{2}}$; or, by multiplying the numerator and denominator by $4-\sqrt{2}$, $x=\frac{1}{14}s(4-\sqrt{2})$.

It is plain that if s had been the difference of the perimeter and diagonal, we should have had $4x-x\sqrt{2}=s$; and therefore, by dividing by $4-\sqrt{2}$, and multiplying the terms of the resulting fraction by $4+\sqrt{2}$, we should obtain the second of the results mentioned in the answer.

* Most of the following solutions are so simple and obvious as not to require to be illustrated by diagrams. Besides, in the cases in which diagrams are not given, the student can readily form them for himself; and he ought to do so.

Exer. 2.

Let $8x$ and $5x$ be assumed to denote the length and breadth. Then, by the question, and by No. 1., Note, page 257., we have $8x \times 5x$ or $40x^2=160$; whence $x=2$; and consequently, $8x=16$ and $5x=10$.

The solution might also be effected by putting x to denote the length or breadth. Thus, if it represent the length, we have $8 : 5 :: x : \frac{5}{8}x$, the breadth; and by multiplying this by x, and putting the product equal to 160, an equation is obtained, the resolution of which will give x.

Exer. 3.

Let the side be represented by $2x$; and since, by the nature of the triangle, and by Euc. I. 47., the square of the perpendicular of the triangle, is equal to the difference of the squares of a side and half the base, we have the square of the perpendicular equal to $(2x)^2-x^2=3x^2$, and, consequently, the perpendicular itself equal to $x\sqrt{3}$. Then, by multiplying this by x, we get $x^2\sqrt{3}$, the area, so that $x^2\sqrt{3}=a$. Hence

$$x^2=\frac{a}{\sqrt{3}}=\frac{a\sqrt{3}}{3}=\frac{3a\sqrt{3}}{9};$$

and, therefore, $x=\frac{1}{3}\sqrt{3a\sqrt{3}}$, the double of which is the side.

Exer. 4.

To give an easy general solution for this problem, since the sum of the length and breadth of the rectangle is plainly $\frac{1}{2}p$, and, consequently, half their sum $\frac{1}{4}p$, we may put x to denote half their difference. Then (ALGEBRA, § 52.) the length will be $\frac{1}{4}p+x$, and the breadth $\frac{1}{4}p-x$; and (Euc. I. 47.) the sum of the squares of these is equal to the square of the diagonal: that is

$$(\tfrac{1}{4}p+x)^2+(\tfrac{1}{4}p-x)^2, \text{ or } \tfrac{1}{8}p^2+2x^2=d^2;$$

whence we readily find $x=\frac{1}{4}\sqrt{(8d^2-p^2)}$; and the length and

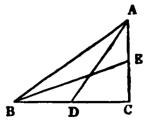
breadth are found by taking the sum and difference of this and $\frac{1}{4}p$.*

The particular answers will be obtained by substituting in the general results thus found, 100 for d and 248 for p. The same may also be effected by assuming the length and breadth respectively equal to $62+x$ and $62-x$, (62 being one fourth of 248), and proceeding as in the general solution.

The solution might be effected in several other ways. Thus, for instance, we might put x to denote the length or breath, and we should have the breadth or length expressed by $\frac{1}{2}p-x$; and the solution would be accomplished by resolving the equation obtained by putting the sum of the squares of these equal to s^2. It may also be solved very easily in the manner pointed out in the note to Exer. 6. ALGEBRA, page 266. That exercise in fact is almost identical with the present one.

Exer. 5.

Let ABC be the required triangle; and, its sides being bisected in D and E, let AD and BE be joined; and, by the question, we have $AD=a$ and $BE=b$. Then an easy solution is obtained by assuming $DC=x$ and $EC=y$, and consequently $BC=2x$ and $AC=2y$; since (Euc. I. 47.) $DC^2+CA^2=AD^2$, and $BC^2+CE^2=BE^2$; which, when expressed in the notation that has been adopted, give equations (1.) and (2.) in the margin. From these (3.) is obtained by addition; (4.) from (1.) by multiplying by 5; (5.) from (3.) and (4.) by subtraction; and (6.) from (5.) by dividing by 15 and by extraction. Equations (7.), (8.), and (9.) are found by a process exactly similar to that employed for finding (4.), (5.), and (6.). The legs of the triangle are found by doubling the values of y



$$x^2+4y^2=a^2 \quad \text{(1.)}$$
$$4x^2+y^2=b^2 \quad \text{(2.)}$$
$$5x^2+5y^2=a^2+b^2 \quad \text{(3.)}$$
$$5x^2+20y^2=5a^2 \quad \text{(4.)}$$
$$15y^2=4a^2-b^2 \quad \text{(5.)}$$
$$y=\sqrt{\frac{4a^2-b^2}{15}} \quad \text{(6.)}$$
$$20x^2+5y^2=5b^2 \quad \text{(7.)}$$
$$15x^2=4b^2-a^2 \quad \text{(8.)}$$
$$x=\sqrt{\frac{4b^2-a^2}{15}} \quad \text{(9.)}$$

* It is plain that if p^2 exceed $8d^2$, the radical $\sqrt{(8d^2-p^2)}$ will be imaginary, and the problem impossible; while, if $p^2=8d^2$, the required rectangle will be a square.

and x found in (6.) and (9.); and the hypotenuse by extracting the square root of the sum of the squares of the results.*

This question may also be solved by means of one unknown quantity in more ways than one. Thus, we might put $DC=x$; then $AC=\sqrt{(a^2-x^2)}$; and an equation which would give the value of x would be found by adding the square of half this to the square of $BC(=2x)$ and putting the sum equal to b^2. The solution given above, however, is perhaps preferable to any other.

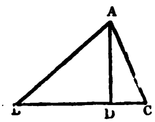
Exer. 6.

The method of solution pointed out in the note to this question is as good as can be desired. We might put, however, the legs equal to x and $s-x$, and the values of x would be had from the equation obtained by putting the sum of the squares of x and $s-x$ equal to h^2.

Exer. 7.

This question may be solved in various ways. There is perhaps none preferable, however, to that which is pointed out in the note, ALGEBRA, p. 266.

Exer. 8.



Let $BD-DC=x$; then, since $BC=BD+DC=2b$, we have (ALGEBRA, § 52.) $BD=b+\frac{1}{2}x$. Now, by a known property of the triangle (Euc. II. 5. cor. 4.), the rectangle under the sum and difference of BD and DC is equal to the rectangle under the sum and difference of BA and AC; that is $2bx=(AB+AC)d$.

* It is plain from the values of x and y, that the question will be impossible, if, of the two quantities a and b, either be greater than the double of the other, as the value of one of those quantities (x and y) would then be imaginary. If b should be equal to $2a$ or to $\frac{1}{2}a$, the triangle would be evanescent, the hypotenuse coinciding with one of the legs. In this case the problem would be, as it were, at the boundary between possibility and impossibility.

Hence $AB + AC = \frac{2bx}{d}$; and therefore (ALGEBRA, § 52.) AB= $\frac{bx}{d} + \frac{1}{2}d$. We have also (Euc. I. 47.) $AB^2 = BD^2 + DA^2$; that is

$$\left(\frac{bx}{d} + \frac{1}{2}d\right)^2 = \left(b + \frac{1}{2}\ \right)^2 + p^2.$$

Hence by actual squaring, by rejecting bx, and by multiplying by 4 and d^2, we get

$$4b^2x^2 + d^4 = 4b^2d^2 + d^2x^2 + 4d^2p^2\ ;$$

and, by transposition,

$$4b^2x^2 - d^2x^2 = 4b^2d^2 - d^4 + 4d^2p^2 = d^2(4b^2 - d^2 + 4p^2)\ ;$$

and hence we get the answer by dividing by $4b^2 - d^2$, and extracting the square root.*

Exer. 9.

The method of solving this question is almost identical with that of the foregoing. Thus, putting BD − DC (last diagram) $= x$, we have $BD = b + \frac{1}{2}x$. We have also (Euc. II. 5. cor. 4.) $2bx = s$ (AB − AC); whence $AB - AC = \frac{2bx}{s}$; and therefore (ALGEBRA, § 52.) $AB = \frac{1}{2}s + \frac{bx}{s}$. Then (Euc. I. 47)

$$\left(\frac{1}{2}s + \frac{bx}{s}\right)^2 = \left(b + \frac{1}{2}x\right)^2 + p^2\ ;$$

the resolution of which equation gives x, the difference of BD and DC.

* This solution affords a striking instance of the advantage of making a judicious choice as to the unknown quantity. Had x been put to denote AB or AC, the solution would have been much more difficult. Much advantage results also, from the use of Euc. II. 5. cor. 4.

Exer. 10.

Here, let $2x$ be assumed to denote the difference of BD and DC. (See the diagram for Exer. 8.) Then (ALGEBRA, § 52.) $BD=b+x$ and $DC=b-x$; and (Euc. I. 47.)

$$AB^2=(b+x)^2+p^2=b^2+2bx+x^2+p^2, \text{ and}$$
$$AC^2=(b-x)^2+p^2=b^2-2bx+x^2+p^2.$$

We then put $2b^2+2x^2+2p^2$, the sum of these latter quantities, equal to s; and the solution is completed by transposing $2b^2$ and $2p^2$, dividing by 2, and extracting the square root.

Exer. 11.

Let, as in the first note in the ALGEBRA, p. 267., the sides be represented by $x-y$, x, and $x+y$, quantities which are evidently in arithmetical progression. Then (Euc. I. 47.)

$$(x-y)^2+x^2=(x+y)^2 \text{ or } x^2-2xy+y^2+x^2=x^2+2xy+y^2.$$

Hence, by contraction, $x=4y$; and therefore the sides, $x-y$, x, and $x+y$, will be $3y$, $4y$, and $5y$. Now, the area being a, we have (by the note, No. 1., ALGEBRA, p. 257.) $3y\times 4y=2a$; whence $y=\sqrt{\frac{1}{6}a}=\frac{1}{6}\sqrt{6a}$; and the answers are found by multiplying this expression successively by 3, 4, and 5.

Exer. 12.

Here, out of the many ways in which this question may be solved, we may represent the sides by $\frac{x}{y}$, x, and xy, which are evidently in geometrical progression. Then, since the area is a, we have $\frac{x^2}{y}=2a$; and also (Euc. I. 47.) we have $\frac{x^2}{y^2}+x^2=x^2y^2$, or (by dividing by x^2, and multiplying by y^2) $1+y^2=y^4$.* The first of these expressions gives $y=\frac{x^2}{2a}$; and by substituting this for

* From this equation we could find the value of y, the common multiplier, which must evidently be independent of the area a.

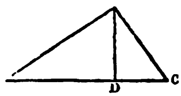
y in the last of them, we get $1+\frac{x^4}{4a^2}=\frac{x^8}{16a^4}$. From this, by multiplying by $16a^4$, and by transposition, we obtain $x^8-4a^2x^4=16a^4$; and by resolving this for x^4 (ALGEBRA, § 154.) we get

$$x^4=2a^2+\sqrt{20a^4}=2a^2+a^2\sqrt{20}=a^2(2+\sqrt{20});$$

and the answer is obtained from this by extracting the fourth root.

Exer. 13.

The properties of the triangle referred to in this question may be established in various ways. Thus, without Algebra, we have (Euc. VI. 8. cor.) BC : BA :: BA : BD; and, by hypothesis, BC : BA :: BA : AC; whence BD and AC, being third proportionals to the same quantities, are equal. We have, also, (Euc. VI. 8. cor.) BC : CA :: CA : CD, or, by what has been proved, BC : BD :: BD : CD; so that (Euc. VI. def. 3.) BC is cut in extreme and mean ratio in D.



The following is a simple algebraic proof of the same. Let $BC=1$, $BA=x$, and $AC=x^2$, which are in geometrical progression. Then, since (Euc. VI. 8. cor.) $CB.BD=BA^2$, and $BC.CD=CA^2$, we have $BD=x^2$ and $DC=x^4$; so that $BC(=1) : BD(=x^2) :: BD(=x^2) : DC(=x^4)$; and $BD=AC$, each being $=x^2$.*

Exer. 14.

Representing the legs by l and l_2, the hypotenuse by h, and the perpendicular by p, we have, by the question,

* Retaining the same assumption, we should have $AD=x^3$, since $BC \cdot AD=BA \cdot AC$, each being double of the area. Hence (Euc. I. 47.) $BD=\sqrt{(x^2-x^6)}=x\sqrt{(1-x^4)}$, and $DC=\sqrt{(x^4-x^6)}=x^2\sqrt{(1-x^2)}$; and therefore $BD=x^2$ and $DC=x^4$, since (Euc. I. 47.) $\sqrt{(1-x^4)}=x$, and $\sqrt{(1-x^2)}=x^2$, which affords another proof.

$$l+l_2=s, \text{ and } h^2-p^2=d.$$

By squaring the members of the first of these, we get

$$l^2+l_2^2+2ll_2=s^2, \text{ or } h^2+2hp=s^2;$$

since (Euc. I. 47.) $l^2+l_2^2=h^2$, and since $2ll_2=2hp$, each of the latter being equal to four times the area of the triangle. From this last result we get $p=\frac{s^2-h^2}{2h}$; and hence the second of the original equations becomes

$$h^2-\left(\frac{s^2-h^2}{2h}\right)^2=d;$$

whence we get

$$3h^4+(2s^2-4d)h^2=s^4,$$

by squaring, clearing of fractions, &c.: and from this h is found by means of § 154., ALGEBRA, p. 147. The particular value of the hypotenuse will be found by taking $s=391$, and $d=69,121$, in the general value; and then the perpendicular p will be computed by means of the value found for it above.

Exer. 15.

Here, employing the same notation as in the last question, we have $l-l_2=5$; whence, by squaring both members, and by substituting h^2 for $l^2+l_2^2$, and $-24h(=-2ph)$ for $-2ll_2$, we get $h^2-24h=25$; and, by resolving this equation, we find $h=25$. Hence h^2 or $l^2+l_2^2=625$, and $2ll_2$ or $24h=600$: and from this, by addition, and by extracting the square root, we get $l+l_2=35$; from which, and from $l-l_2=5$, we get l and l_2 by means of § 52., ALGEBRA, p. 42.

Exer. 16.

Let x and y be put to represent the length and breadth, so that the area may be xy. Then, by the question

$$(x+a)(y+b)-xy, \text{ or } bx+ay+ab=c, \text{ and}$$
$$xy-(x-a_2)(y-b_2), \text{ or } b_2x+a_2y-a_2b_2=c_2$$

From these two equations the values of x and y may be found in any of the ordinary ways. Thus, employing the method given in § 140., ALGEBRA, p. 115., we multiply the first equation by a_2 and the second by a, and we get

$$a_2bx+aa_2y+aa_2b=a_2c, \text{ and}$$

$$ab_2x+aa_2y-aa_2b_2=ac_2;$$

and by taking the members of the latter of these equations from those of the former, we obtain $(a_2b-ab_2)x+aa_2(b+b_2)=a_2c-ac_2$; whence, by transposition and division, we find the value of x. That of y will be found from the same equations by multiplying the first by b_2 and the second by b, taking the difference of the results, &c.

Exer. 17.

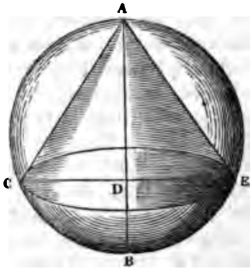
Let CD (see the diagram, ALGEBRA, p. 265.) the radius of the base $=x$. Then, by the question, the altitude AC will be $s-x$; and (by the Note, ALGEBRA, p. 257., Nos. 2. and 3.) the content of the cone is $\frac{1}{3}\pi x^2(s-x)$. Putting this, therefore, equal to c, multiplying by 3, dividing by π, and transposing, we get $x^3-sx^2+3c\pi^{-1}=0$, the general equation for determining x.

By taking $s=40$, $c=3141{\cdot}593$, and $\pi=3{\cdot}141593$, we get the particular equation, $x^3-40x^2+3000=0$; one root of which is readily found by trial, (ALGEBRA, § 189.) to be 10. Then, by dividing x^3-40x^2+3000 by $x-10$, and putting the quotient equal to zero, we get $x^2-30x-300=0$; a quadratic, the resolution of which gives $x=15\pm\sqrt{525}=15\pm5\sqrt{21}$, the remaining roots of the equation.

As is remarked in the ALGEBRA, the two positive roots, 10 and $15+5\sqrt{21}$, afford two distinct solutions of the question, as proposed; the former giving the altitude $(40-x)=30$, and the latter $=25-5\sqrt{21}$. Since the remaining root, $15-5\sqrt{21}$, is negative, if we denote it by $-x'$, we have the altitude $=s+x'$, so that s is the excess of the altitude above the radius x'. Hence, therefore, $5\sqrt{21}-15$ is the radius of the base of the cone required in the following question: — *Given the content of a right cone $=3141{\cdot}593$, and the excess of its altitude above the radius of its base $=40$, to find its dimensions.*

Exer. 18.

Let AB be a diameter of the given sphere; AD the altitude of the required cone, and CD or DE the radius of its base. Then, putting $AD=x$, we have $DB=2a-x$, and (Euc. III. 35.) $CD^2=AD\cdot DB=2ax-x^2$. The content of the cone, therefore, ($\frac{1}{3}\pi.AD.CD^2$, by ALGEBRA, No. 3., Note, p. 257.) is $\frac{1}{3}\pi x(2ax-x^2)$; and by putting this $=c$, performing the actual multiplication, transposing, and dividing by $\frac{1}{3}\pi$, we obtain



$$x^3-2ax^2+3c\pi^{-1}=0, \text{ or } x^3-2ax^2+b=0,$$

b being put to represent $3c\pi^{-1}$.

ELIMINATION, ETC.

(ALGEBRA, p. 271.)

Exer. 1.

Here equation (3.) is obtained by multiplying equation (2.) by x, and taking (1.) from the product. Equation (4.) is found by multiplying (2.) by y, and subtracting (3.) from the product. Hence $x=2y$; and (5.) is obtained by substituting this for x in (2.). The rest of the work requires no explanation.

$$x^3-3yx^2+(3y^2-y+1)x-y^3+y^2-2y=0 \quad \ldots\ldots (1.)$$
$$x^2-2yx+y^2-y=0 \quad \ldots\ldots (2.)$$
$$yx^2-2y^2x-x+y^3-y^2+2y=0 \quad \ldots\ldots (3.)$$
$$x-2y=0 \quad \ldots\ldots (4.)$$
$$4y^2-4y^2+y^2-y=0, \text{ or } y^2-y=0 \quad \ldots\ldots (5.)$$
$$y=0, \text{ or } y=1 \quad \ldots\ldots (6.)$$
$$x=0, \text{ or } x=2 \quad \ldots\ldots (7.)$$

Exer. 2.

In the following operation, equation (3.) is obtained from (1.) and (2.) by subtraction; and (4.) by resolving (3.) for y. The substitution of this value of y in (1.) gives (5.); and (6.) is obtained from (5.) by performing the operations which are indicated, multiplying by 5, and contracting. Hence all that remains to be done is to resolve the equation of the fourth degree in line (6.). Now, by trying (according to § 189.) some of the factors of 30, we readily find that one of the roots is 1, and another 2. Then, dividing the first member of (6.) by x^2-3x+2, the product of $x-1$ and $x-2$, we get $x^2+10x+15$; and, putting this $=0$, and resolving the equation so found, we get $x=-5\pm\sqrt{10}$. Having thus got the four values of x, we have simply to substitute them in equation (4.), to find the corresponding values of y, the doing of which presents no difficulty.

$$yx^2+9x-10y=0 \ldots\ldots\ldots\ldots (1.)$$
$$(y-1)x^2+2x-5y+3=0 \ldots\ldots\ldots\ldots (2.)$$
$$x^2+7x-5y-3=0 \ldots\ldots\ldots\ldots (3.)$$
$$y=\frac{x^2+7x-3}{5} \ldots\ldots\ldots\ldots (4.)$$
$$\frac{x^2(x^2+7x-3)}{5}+9x-2(x^2+7x-3)=0 \ldots\ldots (5.)$$
$$x^4+7x^3-13x^2-25x+30=0 \ldots\ldots\ldots\ldots (6.)$$

Exer. 3.

Here, equation (3.) is found from (1.) and (2.) by subtraction, and (4.) from (3.) by dividing by 12, and slightly changing the form. Now this last equation is satisfied by taking either $x=1$ or $y=1$; and by taking $x=1$, in either (1.) or (2.), we get $y=0$ and $y=-1$; and in like manner, by taking $y=1$ in either of the same equations, we get $x=3$ and $x=2$.

$$x^2+(8y-13)x+y^2-7y+12=0 \ldots\ldots (1.)$$
$$x^2-(4y+1)x+y^2+5y=0 \ldots\ldots\ldots (2.)$$
$$(12y-12)x-12y+12=0 \ldots\ldots\ldots (3.)$$
$$(y-1)(x-1)-0 \ldots\ldots\ldots\ldots (4.)$$

(ALGEBRA, p. 278.)

Exer. 4.

Putting x equal to the given quantity, and finding successively the second, third, and fourth powers (or at once the twenty-fourth power), we get $x^{24}=a^{12}b^4cx$; whence, by dividing by x, and by extraction, we get $x=(a^{12}b^4c)^{\frac{1}{23}}$.

Exer. 5.

Here, by putting x to denote the value of the proposed continued product, and by raising both members of the equation so obtained to the third power, the results to the second power, and those results to the third power (or by finding at once the eighteenth powers of the quantities at first put equal), we get $x^{18}=a^6b^3c\ ^1x$; whence, by dividing by x, and by extraction, we get the answer.

(ALGEBRA, p. 279.)

Exer. 6.

Here the answers are found at once by taking n first equal to 36, and again equal to 50, in the expression, $\frac{1}{6}n(n+1)(2n+1)$. found in the ALGEBRA, Exam. 8., p. 256.

Exer. 7.

The answers to this question are obtained by simply taking n successively equal to 60 and 30 in the expression, $\frac{1}{6}n(n+1)(n+2)$, found at the end of § 267., ALGEBRA, p. 279.

Exer. 8.

Here, if the pyramid were complete, the entire number of balls (ALGEBRA, § 268.) would be $\frac{1}{6}\times 48\times 49\times 97$, or 38,024. The part cut off has in each side of its base 19 balls; and hence it

would contain $\frac{1}{6}\times 19\times 20\times 39$ or 2470 balls; the difference between which and the foregoing result is 35,554 balls, the answer.

Exer. 9.

To solve this exercise according to § 268., ALGEBRA, p. 280., we have, in reference to the first pile, $r=20$ and $n=40$; and therefore the number of balls in that pile, $\frac{1}{6}n(n+1)(3r+2n+1)$, is 38,540.

In the other pile we have $r=30$ and $n=35$; and therefore the required number of balls is $\frac{1}{6}\times 35\times 36\times 161$, or 33,810.

THE END.

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[illegible] multiplying [illegible] the second by [illegible] thus

[illegible] by 6, [illegible] From [illegible] by [illegible] Then, to [illegible] the [illegible] members of [illegible]

[illegible] $y=$ [illegible]

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